ELECTRICITY AND MAGNETISM

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THE MATHEMATICAL THEORY

BY

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PREFACE

In the preparation of this monograph, *Electricity and Magnetism*, deductive methods have been used as far as possible. In general, the appeal is made to reason, though in some cases experimental results have to be used as authoritative; then there are sweeping generalizations which cannot be avoided. Thus the book is rigorous, I hope, in that what is done is clearly stated.

Vector methods are used exclusively, with the exception of that portion of the last chapter which treats of special relativity and its application to electrical theory. Even here, vector methods bring out the results much more sharply; but this would require the development of a four-dimensional vector analysis which would seem unnecessary and undesirable here. Courses in vector analysis are given in most of our universities, in fact, in some of our Eastern technical schools, it is a required course for the undergraduate electrical engineer, so the introduction gives only a brief résumé of the elements with the principal theorems for ready reference.

This book should appeal to a very select class of readers, the physicist, the mathematician, and the electrical engineer who wish to orient themselves quickly in electrical theory. For students with vector analysis as a prerequisite, this book should furnish a three-hour course for one semester. Without the prerequisite, it probably would require three hours throughout the year. In this case, lectures, reading assignments on the text, and a development of the vector analysis when needed could be one method of procedure. The book could be used for a course in vector analysis with applications; it should be inspiring for the student under competent guidance to work out the details of the introductory chapter.

In general, the professional electrical engineer is graduated without being able to read electrical theory. This is a deplorable situation, the theory of electricity certainly should not be a closed book to the good electrical engineer. It would seem feasible to correct this state of affairs by a one-hour course through the year for junior and senior electrical students; this course might consist of a one-hour informal lecture on the text and one hour for problems and discussions, with no further required outside work. This would not cover the material in the text, but enough could be done to arouse an interest and furnish the preparation for subsequent individual study. The problems at the end of the first four chapters are not at all difficult, but they are illustrative, their purpose is primarily to arouse a critical attitude towards the text.

Maxwell, in his theory, prophesied the discovery of electromagnetic waves, the prophecy was confirmed by Hertz in his own laboratory One occasionally hears it stated that there is no ether and there are no waves. This may be just a matter of definition or terminology. All the ether has to do to qualify is to support electromagnetic phenomena; if empty space does this, we still have the ether identified with empty space. The ether is useful in describing electromagnetic phenomena, and it furnishes a useful mechanical language, this is enough to justify the hypothesis Waves are not going out of style; the de Broglie suggestion that waves accompany and govern every moving particle is the point of view of modern quantum mechanical theory, while light phenomena point conclusively to the dual property of a corpuscular wave character with the same de Broglie significance. Then every electromagnetic disturbance in free space is a periodic disturbance which may be described by some form of wave equation.

There are other ways of treating magnetostatics, the classical method is used here Then it is a surprising fact that the classical method is directly in accord with modern

physical ideas. The Ampère-Maxwell point of view is to incorporate into the theory observable measurable quantities, and this is the Heisenberg–Schrodinger–Dirac viewpoint This is probably one reason why classical electrodynamics has persisted under the barrage of modern physics. In the theory of the nature of magnetism, physicists are looking at the atom microscopically, and some more or less satisfactory theories have evolved. No attempt has been made here to treat this subject, the nature of the book would hardly permit it. This monograph is in no way intended as a treatise, but rather as a *path through* the subject of electrical theory. According to the observation of one of the referees, it should be a "big time-saver."

In the preparation of the manuscript, it has been necessary to consult many books and even many original papers, an adequate, though incomplete, bibliography is appended. I am, of course, indebted to many workers, probably to Maxwell most of all. I take this opportunity to express my appreciation to Professor G. Y. Rainisch, who read a large part of the manuscript, for his suggestion that I rewrite the portion on special relativity, which I did with considerable self-satisfaction and to Professor W. W. Denton, who has carefully read all the proofs.

VINCENT C. POOR.

Feb 19, 1931

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ELECTRICITY AND MAGNETISM

INTRODUCTION

VECTOR ANALYSIS

1. **Vector Algebra.**—In the electromagnetic theory two kinds of quantities arise: scalars, which are known to the student of algebra, and vectors. The vector is a geometric line having magnitude, direction, and sense. Certain physical quantities arising will be interpreted as vectors, so for these quantities the following analysis will be applicable. Clarendon type will be used for vectors, and other notations current in the United States will be introduced as needed.

From the parallelogram law we may obtain the commutative and associative laws for vectors

$$\left.\begin{aligned} \mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a} \\ (\mathbf{a} + \mathbf{b}) + \mathbf{c} &= \mathbf{a} + (\mathbf{b} + \mathbf{c}). \end{aligned}\right\} \tag{1}$$

We also define two kinds of products, the scalar product and the vector product, respectively, by the equations:

$$\left.\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos(\mathbf{a}, \mathbf{b}) \\ \text{and} \qquad \mathbf{a} \times \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \sin(\mathbf{a}, \mathbf{b})\mathbf{n} \end{aligned}\right\} \tag{2}$$

where $\mathbf{n}$ is a unit vector perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$ so chosen that $\mathbf{n}$, $\mathbf{a}$ and $\mathbf{b}$ in this cyclic order form a right-handed set. Thus the dot or the cross placed between two vectors indicates, respectively, the scalar or the vector product of those vectors.

If we introduce a rectangular coordinate system so chosen that $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form a right-handed set of unit vectors

along ox, oy, and oz, and if the notation a_x, a_y, a_z, be used for the projections of the vector a on the vectors i, j, k, respectively, then

and
$$\left.\begin{aligned} a &= a_x i + a_y j + a_z k \\ a\cdot b &= a_x b_x + a_y b_y + a_z b_z \\ a \times b &= (a_y b_z - a_z b_y)i + \\ &\qquad (a_z b_x - a_x b_z)j + (a_x b_y - a_y b_x)k \end{aligned}\right\} \tag{3}$$

which may be easily verified from the definitions (2) Also from these definitions the following distributive and commutative laws may be deduced.

$$\left.\begin{aligned} a\cdot(b + c) &= a\cdot b + a\cdot c \\ a \times (b + c) &= a \times b + a \times c \\ a\cdot b &= b\cdot a \\ a \times b &= - b \times a \end{aligned}\right\} \tag{4}$$

The scalar and vector triple products and also the scalar quadruple products are mere applications of (2) and (4), they are, respectively,

$$\left.\begin{aligned} a \times b\cdot c &= a\cdot b \times c = - b \times a\cdot c, \text{ etc.} \\ (a \times b) \times c &= a\cdot cb - b\cdot ca \\ (a \times b)\cdot(c \times d) &= (a\cdot c)(b\cdot d) - (b\cdot c)(a\cdot d) \end{aligned}\right\} \tag{5}$$

The first of (5) is the volume of the parallelopiped formed by the three vectors

One other important algebraic form expresses the fact that every vector d may be expressed linearly in terms of three arbitrary noncomplanar vectors, explicitly,

$$(a \times b\cdot c)d = (b \times c\cdot d)a + (c \times a\cdot d)b + (a \times b\cdot d)c$$

VECTOR CALCULUS

2. The Differential.—In the study of scalar and vector point functions or functions of a parameter, the vector calculus naturally arises The *differential* of a function, scalar or vector, may be given by the definition:

$$df = \underset{h \to 0}{\text{Limit}} \frac{f(t + hdt) - f(t)}{h}. \tag{6}$$

If t is a scalar parameter, a division by dt gives the ordinary differential quotient of the function. If t is the point P or the radius vector r the same definition furnishes the differential of the function. If we write

$$r = P - O$$

where the point O is an arbitrarily chosen origin, then by dP we mean an arbitrary displacement of the point P, a vector, so that

$$dr = \underset{h \to 0}{\text{Limit}} \frac{(P - O) + hdP - (P - O)}{h} = dP$$

follows directly from our definition Referred to rectangular cartesian coordinates, since $r = xi + yj + zk$

$$dr = dP = dxi + dyj + dzk. \tag{7}$$

We will use indiscriminately dr, dP, δP, or some such symbol, to mean the arbitrary displacement of a point

From the above important definition the following theorems may easily be proved

$$\left.\begin{array}{ll} (a) & dcu = cdu \\ (b) & d\phi u = \phi du + ud\phi \\ (c) & d(u + v + w) = du + dv + dw \\ (d) & d(u \cdot v) = u\ dv + v\ du \\ (e) & d(u \times v) = u \times dv + (du) \times v. \end{array}\right\} \tag{8}$$

Greek letters will be used, in general, as scalar functions, as in (8b).

Certain indefinite integration formulae may be directly deduced from these, among which

$$\int u\, du = \frac{u^2}{2} + c \tag{9}$$

is probably the most frequently used

3. Flux and Flow.—There are two definite integrals which are of fundamental importance The flux integral

$$\int_\sigma u \; n d\sigma$$

where u is a vector point function and n is a unit vector normal to the surface element $d\sigma$. Since $u \cdot n$ is the projection of the vector u on the normal to the surface element, it is a scalar, so the integral is an ordinary surface integral taken over the surface σ. This integral is frequently referred to as the flux of the vector u through the surface If u is the velocity of a liquid, the flux integral furnishes the quantity of liquid crossing the surface per unit time if the density is unity. In our coordinate system the vector

$$d\sigma n = dydz i + dzdx j + dxdy k$$

so that

$$\int_\sigma u \cdot n d\sigma = \int u_x dydz + \int u_y dzdx + \int u_z dxdy,$$

or the flux integral breaks up into the sum of three integrals taken over the projected areas in the coordinate planes.

The flow integral

$$\int_c u \; dr$$

where the displacement dr is taken along the curve c, is a line integral along a curve. If coordinates are introduced it is reducible to three line integrals along the coordinate axes, or

$$\int_c u \cdot dr = \int u_x dx + u_y dy + u_z dz.$$

If the vector u defines a field of force, the flow integral is then the work done by the force in moving a mass along the curve. If the curve c is a closed contour the flow integral is called the circulation

These integrals, of course, imply certain integrability conditions on the functions, and restrictions on the surface σ and the curve c

4. **Grad ϕ, div u, curl u.**—The *gradient* of a scalar point function ϕ may be defined by the equation

$$d\phi = \text{grad } \phi \; dr \tag{10}$$

and since

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k.$$

Thus grad ϕ is a vector normal to the equipotential surface $\phi = c$. Its modulus is the square root of the sum of the squares of the partial derivatives of ϕ. If the "nabla" operator

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$

be introduced, we may write symbolically

$$\text{grad } \phi = \nabla\phi$$

The *divergence* of a vector, u, written div u, may be defined by the equation

$$\underset{\tau \to 0}{\text{Limit}} \frac{\int u \; n d\sigma}{\tau} \equiv \text{div } u. \tag{11}$$

The numerator is the flux integral over the closed surface σ bounding the region τ, the surface is contracted about a point P in τ, giving in the limit the divergence of the vector u at the point P. Out of the definition (11) Gauss's theorem, the flux divergence theorem follows:

$$\int_\sigma u \cdot n \; d\sigma = \int_\tau \text{div } u d\tau. \tag{12}$$

In fact the definition itself may be looked upon as the application of Gauss's theorem to a volume element. When (12) is applied to the cuboid $dxdydz$, div u comes out in the coordinate form

$$\operatorname{div} u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z},$$

or using the nabla

$$\operatorname{div} u = \nabla \cdot u.$$

If σ is a contour bounding a surface σ, and if n is the unit normal to σ at a point P of the surface, then we may write the defining equation

$$\underset{\sigma \to 0}{\text{Limit}} \frac{\int_\sigma u \cdot dr}{\sigma} = w \cdot n, \tag{13}$$

where it is understood that in passing to the limit the contour σ is contracted over the surface around the point P. The vector w here defined is the rotational or curl of the vector u, written curl u. This definition is also the elementary form of Stokes's theorem

$$\int_\sigma u \cdot dr = \int_\sigma \operatorname{curl} u \cdot n d\sigma \tag{14}$$

which may be directly deduced from (13). It is sometimes called the flow curl theorem, it expresses the relation that the flux of curl u through the surface σ is equal to the flow of u around its boundary. If (13) is applied to the parallelogram $dxdy$ at P, the projection on oz

$$\operatorname{curl} u \cdot k = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$$

will result. The other components or projections may be obtained in a similar way; we may thus write

$$\operatorname{curl} u = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) i + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) j + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) k$$

or in terms of the nabla

$$\operatorname{curl} u = \nabla \times u.$$

Associated with every scalar field there is a derived vector field, the gradient of the scalar point function. Associated with every vector field there are two derived fields, the scalar field of the divergence of the vector, and the vector field, the curl of the vector.

By using coordinates, the nabla operator, or some other scheme, the following sets of theorems may be proved:

$$\left.\begin{aligned} \operatorname{grad} c &= 0 \\ \operatorname{div} a &= 0 \\ \operatorname{curl} a &= 0 \end{aligned}\right\} \tag{15}$$

where c and a are constants;

$$\left.\begin{aligned} \operatorname{grad} (\phi + \psi) &= \operatorname{grad} \phi + \operatorname{grad} \psi \\ \operatorname{div} (u + v) &= \operatorname{div} u + \operatorname{div} v \\ \operatorname{curl} (u + v) &= \operatorname{curl} u + \operatorname{curl} v \\ \operatorname{grad} c\phi &= c \operatorname{grad} \phi \\ \operatorname{div} cu &= c \operatorname{div} u \\ \operatorname{curl} cu &= c \operatorname{curl} u \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} &(a)\ \operatorname{div} \phi u = \phi \operatorname{div} u + u \cdot \operatorname{grad} \phi \\ &(b)\ \operatorname{curl} \phi u = \phi \operatorname{curl} u - u \times \operatorname{grad} \phi \\ &(c)\ \operatorname{div} u \times v = v \cdot \operatorname{curl} u - u \cdot \operatorname{curl} v. \end{aligned}\right\} \tag{17}$$

5. Green's Theorem. By substituting ψ grad ϕ for **u** in Gauss's theorem (12), we will obtain the first form of Green's theorem

$$\int_\tau \psi \Delta \phi d\tau - \int_\sigma \psi \operatorname{grad} \phi \cdot n d\sigma = - \int_\tau \operatorname{grad} \phi \cdot \operatorname{grad} \psi d\tau \tag{18}$$

where the symbol, $\Delta =$ div grad which also may be written as

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

the Laplacian operator. The second form of Green's theorem may be deduced from the first by interchanging

ϕ and ψ in (18) and subtracting the result from (18). The final result is

$$\int_\tau (\psi\Delta\phi - \phi\Delta\psi)\,d\tau = \int_\sigma (\psi \operatorname{grad} \phi - \phi \operatorname{grad} \psi)\cdot n\,d\sigma. \tag{19}$$

These theorems of Gauss, Stokes and Green, are some of the outstanding theorems of analysis.

6. The Operator Δ'. We shall have occasion to use the operator

$$\Delta' = \operatorname{grad} \operatorname{div} - \operatorname{curl} \operatorname{curl} \tag{20}$$

which operates on vectors and evidently obeys the distributive law of algebra. In connection with this operator there arises the fundamental theorem:

$$\Delta'(\phi a) = (\Delta\phi)a \tag{21}$$

The absolute geometric proof of this theorem is beyond the scope of the elements of vector analysis, but the theorem may be easily demonstrated by resorting to coordinates. If one writes out the right member of (20) in rectangular coordinates, one will obtain the Laplacian operator, which is the coordinate form for Δ. Thus one may keep or drop the superscript *ad libitum*. Or if the x-axis is taken parallel to a, which vector may be taken as a unit vector without affecting the generality of (21), this theorem

$$\Delta(\phi i) = (\Delta\phi)i \tag{22}$$

will come out directly in coordinate form if one will take the trouble to write out the left member of this last equation.

CHAPTER I

ELECTROSTATICS

1. Introductory.—The subject of physics has been revolutionized during the past twenty-five years. Newtonian mechanics has become a first approximation in the light of the Einstein Relativity Theory. This is also true of classical electrodynamics

Though we still try to interpret electricity and electrical phenomena in a mechanical way hypotheses of "quanta" have entered electrodynamic theory through radiation phenomena, which are electromagnetic in character. The "one fluid" and the "two fluid" theories of electricity have been supplanted by the atomistic theory Even the "field" theory of Faraday and Maxwell which displaced the "action at a distance" theory has ceased to be so important. In fact, physicists are investigating now the relation of electricity to matter instead of properties of electricity as formerly.

Physical experiments of recent years point conclusively to the fact that electricity is atomistic in structure. The electron, a negatively charged corpuscle, of which all other charges are integral multiples, has been isolated and its charge determined. The electron is believed to be a purely electronic charge possessing inertia, but entirely dissociated from matter. Its mass at rest is roughly one-eighteen hundredth that of the hydrogen atom, the lightest known atom. Strange to say, this mass, which we shall term its electromagnetic mass, is a function of its velocity and increases without limit as its velocity approaches the velocity of light

On the other hand, the positively charged particle is

atomistic in size and its charge has never been dissociated from matter, its mass is many times that of the electron. We will adopt the term *proton* for the particle atomistic in size which carries a positive charge equal to the charge on the electron, to single it out from the *ion* which may be a positively or a negatively charged atom, group of atoms or molecules

The *cathode rays*, the *canal rays* and the rays emitted by *radioactive* substances like uranium and radium are in general electrically charged particles called α and β particles. The former, constituting the canal rays, are through spectroscopy known to be helium atoms each carrying a positive charge $2e$ where $-e$ is the charge on the electron. The β and γ rays constitute the cathode rays; the β particles are electrons while the γ rays are ether waves. The velocities with which the particles are ejected by some of the radioactive substances and the extremely slow chemical change in the substance indicate an enormous amount of energy stored up in the atom. The velocities of the more ponderable α particles are relatively small when compared with the velocities of the electrons, which in some cases move with velocities approaching that of light

The results of electrolysis, the phenomena of radioactivity and spectroscopic analysis point to the conclusion that the atoms of all matter are each made up of a ponderable proton or positively charged nucleus, around which in their respective orbits move a sufficient number of electrons to make the atom electrically neutral. Using the Rutherford atom which pictures the atom as such a planetary system, Niels Bohr has constructed a periodic table of the ninety odd elements. The number of natural unit charges, e, on the nucleus is just equal to the number of the element in the periodic table. This number is called the "atomic number" of the element. The first element is hydrogen, with a single unit charge, e, on the nucleus and one electron, the second is helium with a

charge, $2e$, on the nucleus, and in its normal state with two electrons forming its planetary system. Each element has a characteristic Röntgen or X-ray spectrum which distinguishes it from the other elements.

2. The Distribution of Electric Charge.—It may be convenient at times to consider an electric charge as a point charge, just as in mechanics we use the idea of a mass point. This point charge may include many electrons and protons or just a single electron. On the other hand, we may wish to characterize the distribution of the electronic charge; we might then conceive the electron as occupying a small volume with electric volume density ρ, a continuous point function of the region occupied by the electron. The total charge e would then be given by the integral

$$e = \int \rho d\tau$$

taken throughout the volume of the electron. To avoid discontinuities, we might assume a transition layer at the surface where ρ decreases rapidly to zero; the electric charge would thus appear only as a modification of the *ether*, a substance which we will characterize only as pervading all space and capable of supporting electric phenomena.

Had we considered the electronic charge to be distributed over the surface of the electron with surface density ω then the integral

$$e = \int \omega d\sigma$$

extended over the surface of the electron would furnish the charge, the volume density ρ being zero in the interior would imply the existence of *ether*, or as we may term it, free space in the interior of the electron.

In electro-statics, we shall use the idea of the discreet charge and frequently we shall resort to the idea of a con-

tinuous distribution of charge, replacing the actual electric density by its average value Because of the coarseness of our measuring instruments, the effects of large numbers of ions only are measurable

To arrive at the idea of average values, we divide a region up into elements, macroscopically small, so small that the distribution of charge as measured by any measuring instrument would appear uniform, then the average value $\bar{f}$ of every function f over the region τ is defined by the equation

$$\bar{f} = \frac{1}{\tau}\int f d\tau \ *$$

f being a scalar or vector point function.

The average density $\bar{\rho}$ then becomes

$$\bar{\rho} = \frac{1}{\tau}\int \rho d\tau$$

where ρ is the variable density of electrification, being zero in free space as previously noted. If N is the number of positive unit charges e in τ and n the number of electrons this definition gives

$$\bar{\rho} = \frac{(N - n)e}{\tau}.$$

For matter in its normal state $N = n$ so that $\bar{\rho}$ is zero for ordinary matter. We shall use ρ for the average density unless there is ambiguity, then we shall use $\bar{\rho}$

3. Priestley's Law.—We have discriminated between two kinds of electricity by positive and negative signs. This distinction is not just a matter of sign, but as has been noted, it is qualitative as well.

It has been known for a long time that electrification can be produced by friction, possibly by rubbing an ebonite rod with a woolen cloth Equal amounts of each kind of

* If f were a function of the time and $d\tau$ a time interval, this equation would serve as a definition for time averages.

electricity will be produced, a positive charge on the cloth and a negative charge on the ebonite rod

That like charges repel and unlike charges attract each other is a fact determined by a very simple experiment Experiments also point to the conclusion that this force mutually exerted by charged bodies on each other is a *Newtonian* force. To formulate the law exactly, we conceive of two point charges q_1 and q at a distance r apart then:

The force acts in the line joining the particles with a magnitude proportional to the product of the charges and the inverse square of their distance apart.

This is Priestley's law if named for its first discoverer.*

If we write F for the force exerted by q_1 on q we may formulate the law mathematically by the equation

$$F = \frac{Aq_1q}{r^2}\frac{r}{r} \tag{3.1}$$

where $\frac{r}{r}$ is a unit vector lying in the line joining the charges and directed away from the charge q_1; the proportionality factor, A, depends on the units in terms of which F, r, q_1 and q are measured. The law as formulated makes the repulsive force positive and the attractive force negative; the sense is taken care of automatically by the signs of the charges. The force, F, so chosen has the direction and sense in which a positive charge will be driven.

The charges, q_1 and q, are made up of electronic charges $\pm e$. If N_1 and N are the number of positive unit charges, e, and n_1 and n the number of electrons in q_1 and q, respectively, then

$$\frac{q_1q}{r^2} = \frac{(N_1 - n_1)(N - n)e^2}{r^2}$$

$$= \frac{N_1N + n_1n - N_1n - n_1N}{r^2}e^2.$$

* Whittaker "History of the Theory of the Ether and Electricity." 1910, pp 50, 56. O W. Richardson. "The Electron Theory of Matter," 1914, p. 12.

We see that the resultant force is due to the mutual repulsions of N_1 and N positive unit charges n_1 and n electrons and the mutual attractions of N_1 and n, and n_1 and N oppositely charged unit particles, as we should expect

For a system of n point charges Priestley's law takes the form

$$F = q\sum_{i=1}^{n}\frac{q_i}{r_i^2}\frac{\mathbf{r}_i}{r_i}$$

r_i being the distance from the charge q_i to the charge q. The force F is the resultant of the forces due to the n charges acting on q For a distribution of charge with continuous volume density ρ, an exact formulation of this law may be made through the fundamental law of the integral calculus

4 **The Common Electrostatic Unit Charge.**—Besides the natural unit charge carried by the electron we will adopt a common electrostatic unit charge to simplify some of the formulae If the centimeter is taken as unit length and the dyne as unit force, then we may define unit charge as *that charge which repels a like charge at unit distance in vacuo with a force of* 1 *dyne.* The system of units based on this definition of unit charge is the *Common Electrostatic System,* abbreviated (c. e. s.), unless otherwise stated, we shall use this system. The rational electrostatic system which is sometimes used defines the unit charge in a similar way, but the like charges to be unit charges must repel with a force equal to $\frac{1}{4\pi}$ dynes.

In our chosen system, Priestley's law for free space becomes

$$F = \frac{q_1 q}{r^2}\frac{\mathbf{r}}{r} \tag{4.1}$$

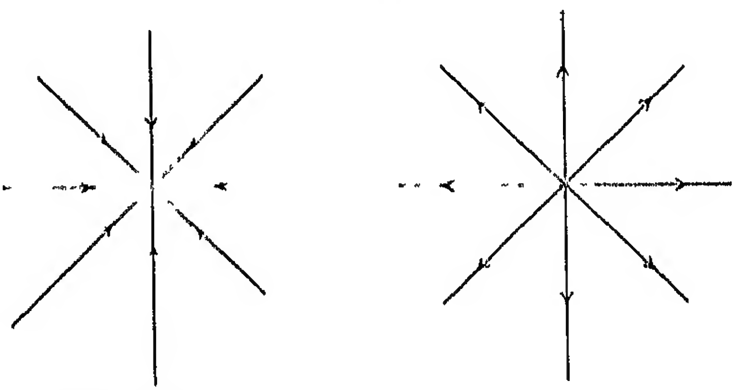
for the charge q_1, acting on the charge q.

5. The Electric Field Intensity. If F is the force acting on a charge, q, at a point, P, the equation

$$F = qE \tag{5.1}$$

defines a new vector, E, a force intensity sometimes called the *electric force, voltivity*, but most properly called the *electric intensity*. It has the direction and sense of the vector F. The intensity E being force per unit charge is independent of the charge q, but depends on the position of the point P and the distribution of electrification. For the point charge q_1, equation (1.1) for free space gives

$$E = \frac{q_1}{r^2}\frac{r}{r} \tag{5.2}$$



Field of the Electron　　　　Field of the Proton.

Fig. 1

as the intensity at the point P. If the charge q_1 is an electron, the field of the vector E will be a radial field, with the sense of the vector E always towards the electron. If q_1 is a proton, the field of the intensity will again be a radial field. The vector lines of the vector E, the lines of electric intensity, will be the half rays directed away from the proton (Fig. 1). Here, as always, the positive sense of the intensity E, like that of the force F, is the sense in which a positive charge would be driven.

The lines of intensity are in general not so simple, they may be any whatever, depending on the distribution of electrification. Whatever this distribution of charge, equation (5.1) defines an electric intensity, E, which characterizes the force F in such a way that if a charge q be brought into the field without disturbing the field, the force F will be given by equation (5.1) If we draw the totality of vector lines of the vector E through the boundary of the surface element $d\sigma$, we will obtain an elementary tube of intensity; the intensity E is tangent at every point of the lateral surface The whole field of intensity may be mapped out by such tubes.

Surrounding every electric charge, there is a peculiar condition characterized by the vector E, or the lines of intensity It is difficult to conceive of a change in state of empty space—hence the ether hypothesis We may then make the concept concrete as a strained state of the ether surrounding the charge Priestley's law furnishes the basis for the "action at a distance" hypothesis, but most people, like Faraday, prefer the ether hypothesis for interpretation purposes

6. The Electrostatic Potential is a scalar point function, the negative gradient of which is the electric intensity E This potential function, which we designate by ϕ, evidently depends on the distribution of charge. We shall show that for the point charge q_1, in free space,

$$\phi = \frac{q_1}{r}$$

r being the distance from the point M_1, where q_1 is situated, to some arbitrary point P.* The equipotential or level surfaces

$$\phi = c$$

* In rectangular cartesian coordinates

$$r = \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2}$$

are concentric spheres with centers at the point M_1, the normal derivative will thus be the derivative in the r direction, or

$$-\operatorname{grad}\phi = -\left(\frac{d}{dr}\frac{q_1}{r}\right)\frac{r}{r} = \frac{q_1}{r^2}\frac{r}{r} \tag{6.1}$$

which agrees with equation (5 2), thus ϕ satisfies the above definition for a potential function

The function

$$\phi = \frac{q_1}{r} + c$$

would have served just as well, since its negative gradient also gives the intensity E. But by assigning a particular value to the constant c, any whatever, then and only then, can we speak of the potential at a point. We have chosen c so that the potential ϕ vanishes at infinity; this makes c identically zero, and $\phi = \frac{q_1}{r}$ the Newtonian potential function.

For two point charges, q_1 and q_2, situated at points M_1 and M_2, respectively, at distances r_1 and r_2 from the point P, the potential at the point P will be the sum of the potentials of each charge taken separately, or

$$\phi = \phi^{(1)} + \phi^{(2)} = \frac{q_1}{r_1} + \frac{q_2}{r_2}$$

This follows from the fact that the sum of the intensities E_1 and E_2 at the point P, due to the charges q_1 and q_2, respectively, is the resultant intensity E, and that the gradient of a sum is the sum of the gradients; thus

$$-\operatorname{grad}\phi = -\operatorname{grad}\phi^{(1)} - \operatorname{grad}\phi^{(2)}$$

which is identically

$$E = E_1 + E_2$$

the level surfaces

$$\phi^{(1)} = c_1, \quad \phi^{(2)} = c_2$$

being again concentric spheres, centers M_1 and M_2, respec-

tively. Since this same argument applies to n point charges, we have for the potential at some point, P, due to the n charges

$$\phi = \sum \phi^{(i)} = \sum_{i=1}^{n} \frac{q_i}{r_i}$$

r_i being the distance from the point P to the point M_i, where the point charge q_i is situated

It may be observed that this result and the argument is true whatever the law of force

For a distribution of charge with a continuous volume density ρ, we replace q_i by $\rho_i d\tau$, and pass to the limit which, according to the fundamental law of the integral calculus, gives

$$\phi = \int \frac{\rho d\tau}{r} \tag{6.2}$$

The negative gradient of this function is the electric intensity E due to the continuous distribution of charge, since the equation

$$E = -\operatorname{grad} \phi \tag{6.3}$$

holds when ϕ is determined by n elements whatever the n

The general form will evidently be

$$\phi = \int \frac{\rho d\tau}{r} + \int \frac{\omega d\sigma}{r} \tag{6.4}$$

if a surface distribution of charge be included. The distance r from the element $d\tau$ at a point M to the point P is a function of the two points M and P Since M is the point of integration the potential ϕ depends on the distribution of electrification, and is a function of the point P alone.

The surface density is not just a mathematical fiction; but it is the density of a real continuous surface distribution of electricity, as far as can be determined by our measuring instruments. To define it more carefully as Maxwell does, we assume a stratum or surface layer of electric volume density ρ, thickness ν, then let ρ increase

without limit as ν approaches zero in such a way that the product $\nu\rho$ remains finite. The limit of $\nu\rho$ as ν approaches zero is the surface density, ω.

For points P exterior to the charged body, the potential and all of its derivatives are continuous single valued functions of the point P. But for points interior to the volume distribution of electrification, the integrand of the integral, $\int \frac{\rho d\tau}{r}$, becomes infinite to the first degree when the point of integration coincides with the point P, yet this integral, its gradient, and the divergence of its gradient are convergent integrals at such points.

The last and least likely case is treated in Art. 11, where it is shown that

$$\Delta\phi = -4\pi\rho.$$

Thus $\Delta\phi$ is discontinuous where ρ is discontinuous.

To prove the convergence of ϕ at a point P of the charged medium, we divide τ into τ_1 and τ_0, where τ_0 is the volume of a small sphere, radius, a, center at P. Then

$$\int_\tau \frac{\rho d\tau}{r} = \int_{\tau_1} \frac{\rho d\tau}{r} + \int_{\tau_0} \frac{\rho d\tau}{r}.$$

The integral over τ_1 and its gradient are evidently convergent, since the point M of integration never coincides with the point P. In the integral over τ_0 we take the volume element $d\tau$, equal to $4\pi r^2 dr$. This integral then becomes

$$4\pi \int_0^a \rho r dr$$

and it is a proper integral. Hence ϕ is a convergent integral. The proof that grad ϕ is a convergent integral will be left as an exercise.

If we use r for the radius vector, and the dot $(\cdot)$ to indicate the scalar product of two vectors, then from the equation

$$d\phi = \text{grad}\ \phi \cdot dr$$

which defines grad ϕ, we can obtain another form for the electric potential function ϕ. Since $E = -\text{grad}\,\phi$ we will have

$$-\int_{P_0}^{P} E\, dr = \int_{P_0}^{P} \text{grad}\,\phi\, dr = \int_{P_0}^{P} d\phi$$

or

$$\phi - \phi_0 = -\int_{P_0}^{P} E\, dr$$

where the integration is to be taken along some path from the point P_0 to the point P. But from this result we see that the line integral is independent of the path of integration; it depends on the terminal points only; it is therefore a scalar point function. This line integral is called the *electromotive force*, or for static fields, the *voltage*, *difference of potential*, or the *fall of potential* between the points P and P_0. To completely identify the function ϕ, we may impose the condition that it vanish at infinity This will be equivalent to taking P_0 at infinity, ϕ_0 then vanishes and we have

$$\phi = -\int_{\infty}^{P} E\, dr \tag{6 5}$$

as the potential at the point P The potential at P_0, some point in finite space, being

$$\phi_0 = -\int_{\infty}^{P_0} E\, dr,$$

we have the difference

$$\phi - \phi_0 = -\int_{\infty}^{P} E\, dr + \int_{\infty}^{P_0} E\, dr.$$

which, upon interchanging the limits of the last integral, reduces to the original form

$$\phi - \phi_0 = -\int_{P_0}^{P} E\, dr$$

for the difference of potential

If a free positive charge would be driven from the level surface through P to the level surface through P_0, then P is said to be of higher potential than P_0, or $\phi - \phi_0$ actually represents a fall of potential. *Positive charges are always driven from points of higher to points of lower potential*

Since qE represents force, the integral in equation (6.5) represents the negative of the work done in transporting a unit charge from infinity to the point P. This fact will be seen to be more significant in the next article.

7. The Potential Energy of a Charge in an Electric Field.—Suppose a particle of mass m and charge q is acted on by a system of electrically charged bodies with a force F, the equation of motion of the particle is then

$$m\ddot{r} = F. \tag{7.1}$$

If we multiply this equation scalarly by $\dot{r}dt$ we will have

$$m\dot{r}\,\ddot{r}dt = F\cdot dr.$$

Upon integrating this equation from $t = t_0$ to $t = t$, regarding m as constant, we get

$$\tfrac{1}{2}m\dot{r}^2 - \tfrac{1}{2}m\dot{r}^2{}_0 = \int_{P_0}^{P} F\cdot dr. \tag{7.2}$$

The left member of the equation is the difference of the kinetic energies of the particle at the times t and t_0, while the right member, a line integral, is the work done in transporting m from a certain position P_0, at the time t_0, to a final position P at the time t. If the force F is independent of the time and equal to the negative gradient of a scalar point function W, the right member of (7 2) becomes

$$-\int_{P_0}^{P} \operatorname{grad} W\, dr = -\int_{P_0}^{P} dW = W_0 - W. \tag{7.3}$$

In this case the line integral

$$\int F\, dr$$

is independent of the path, depending only on the initial

and final positions of the particle The function W is called the potential energy of the particle. The potential energy is thus seen to be a function of position. If we call the kinetic energy T at the time t, and T_0 at the time t_0, we shall have

$$T - T_0 = W_0 - W;$$

or

$$T + W = T_0 + W_0$$

that is, the total energy is conserved throughout the motion. The system of forces acting is thus called a *conservative system* It is always so that when such a potential energy function exists for the applied forces, the system of forces is *conservative*. If we write from equation (7.3)

$$W = -\int_{P_0}^{P} F \cdot dr + W_0$$

we see that we may define the potential energy of the particle as *the negative of the work done by the forces in some arbitrary displacement.* We can make the potential energy unique by determining it from a standard state. If as in the potential function we take infinity as the position of zero potential energy, P_0 will be a point at infinity, and W_0 will be zero, so that

$$W = -\int_{\infty}^{P} F \cdot dr$$

satisfies these conditions. We may now define the potential energy of the particle as *the negative of the work done by the forces in displacing the particle from infinity to the point* P, the work being independent of the path. Also, since

$$F = qE$$

$$W = -q\int_{\infty}^{P} E \cdot dr.$$

But the potential

$$\phi = -\int_{\infty}^{P} E \cdot dr$$

therefore

$$W = q\phi. \tag{7.4}$$

Thus the potential energy of the particle of mass m, charge q, at a point P, due to a system of charged bodies, is the charge q, multiplied by the potential at the point P, due to those charges—a very convenient result.

8. The Mutual Potential Energy of an Electric Field.—To obtain the total potential energy due to n electrically charged particles, we would have to compute the negative of the work done in transporting the system of particles from a state of infinite dispersion at infinity along some set of paths to their final position. We will call the potential function due to the charge q_i, $\phi^{(i)}$, and its potential energy in its final position P_i due to the other particles in the field w_i. We will then bring the charges up in order as numbered. First the potential energy w_1 of the charge q_1 in its final position will be zero, since there are no charges in the field and the force acting is therefore zero. The energy

$$w_2 = q_2\phi^{(1)}$$

according to the last article, where $\phi^{(1)}$ is the potential due to the charge q_1. The potential energy

$$w_3 = q_3(\phi^{(1)} + \phi^{(2)})$$

since the potential function at the point P_3 is the sum of the potentials due to the charges q_1 and q_2, respectively. Also

$$w_4 = q_4(\phi^{(1)} + \phi^{(2)} + \phi^{(3)})$$

and for the general law we will evidently have

$$w_j = q_j\sum_{i=1}^{j-1}\phi^{(i)} \tag{8.1}$$

with the condition that w_1 is zero. If now we reverse the order in which the particles were brought up from infinity, we will have first

$$w_n = 0$$
$$w_{n-1} = q_{n-1}\phi^{(n)}$$
$$w_{n-2} = q_{n-2}(\phi^{(n)} + \phi^{(n-1)})$$
$$w_{n-3} = q_{n-3}(\phi^{(n)} + \phi^{(n-1)} + \phi^{(n-2)})$$

and the general law will again be evident, or

$$w_j = q_j \sum_{i=j+1}^{n} \phi^{ij} \tag{8.2}$$

with the condition that $w_n = 0$. We will obtain twice the total potential energy by summing (8.1) and (8.2) with respect to j from 1 to n, and adding, thus:

$$2W = \sum_{j=1}^{n} q_j \sum_{i=1}^{j-1} \phi^{ij} + \sum_{j=1}^{n} q_j \sum_{i=j+1}^{n} \phi^{ij}$$

$$= \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{j-1} \phi^{ij} + \sum_{i=j+1}^{n} \phi^{ij} \right),$$

or

$$W = \tfrac{1}{2} \sum_{j=1}^{n} q_j \phi_j \tag{8.3}$$

where ϕ_j is the sum of the potentials at the point P_j, due to all charges excepting the charge q_j; the potential ϕ^{jj} does not appear in the last parentheses.

Equation (8.3) is then the total or mutual potential energy of the system of n particles.

To obtain the mutual potential energy of a continuous volume distribution of electrification of density ρ, we replace q_j by $\rho_j d\tau$ and pass to the limit; equation (8.3) then becomes

$$W = \tfrac{1}{2} \int_\tau \phi \rho d\tau$$

where ϕ is the potential at the point M of the element $d\tau$, due to all the charges excepting the charge of the element $d\tau$. If a surface distribution of surface density ω be included, the mutual potential energy evidently is then

$$W = \tfrac{1}{2} \int_\tau \phi \rho d\tau + \tfrac{1}{2} \int_s \phi \omega d\tau \tag{8.4}$$

where according to the notation the first integral will be taken throughout all charged volumes and the second integral over all surface distributions of electricity. It

may be observed again that the result here obtained for the mutual potential energy is independent of the law of force, it is implied, however, that there exists a potential energy function for the forces which vanishes at infinity.

9. Induced Electrification.—Experiments made by Faraday and repeated by Cavendish and Maxwell point conclusively to the fact that an electric charge cannot persist on the interior of a conductor; in fact, this statement may be taken as the definition for a conductor of electricity. The statement that an electric charge cannot persist on the interior of a conductor, though paradoxical, only serves to emphasize the fact that we are using average values, that our measuring instruments are not capable of analyzing the atom. We are only interested in an excess of electricity in a macroscopic element, one containing many atoms or molecules.

If a positively charged body be suspended by some insulating material inside a hollow similarly *insulated* or suspended conductor, the inner surface of the conductor will be charged negatively, and the outer surface positively. The conductor in this case is said to be charged by induction; the induced charges are each quantitatively equal to the charge producing the excitation. Had the original charge been negative, the signs of the induced charges would have been reversed.

The phenomena of induction is easily explained on the electron theory In this theory, it is assumed that electrons are present in conductors in great numbers, that many of them are free, or so loosely bound to the positive nuclei of the atoms that they are easily displaced by a very weak electric field. The electrons are only sufficient in number, of course, to make the macroscopic element of the conductor electrically neutral.

In the case under consideration, the lines of intensity have their sources in the charge producing the excitation and terminate in charges outside the body Under the influence of this field of intensity which permeates the

conductor the free electrons are drawn to its inner surface until equilibrium is established, leaving an excess of positive charges on the outer surface. The lines of intensity with sources in the induced positive charge and terminating on the induced negative charge must then be just equal and opposite to the lines of intensity whose sources are the charges producing the excitation, otherwise the electrons would not be in equilibrium. Thus the resultant intensity is everywhere zero in the interior of the conducting medium.

We can use this result to show that the surfaces of a conductor are equipotential surfaces. In the integral

$$\phi_2 - \phi_1 = -\int_{P_1}^{P_2} E \cdot dr$$

we can take P_1 and P_2 in the same surface of the conductor and the path of integration through the conducting medium. Accordingly, E is identically zero at every point of the path of integration, hence $\phi_1 = \phi_2$, or ϕ is constant on the surface of the conductor. This conclusion is also strengthened by the facts that the charges on the surface are in equilibrium and that the fields of intensity inside and outside a hollow conductor are entirely independent of each other. Thus the lines of intensity terminate on the surface of a conductor.

10. Gauss's Electric Flux Theorem.—For a system of charges embedded in the ether we may prove the following theorem

The flux of the electric intensity **E** *through a closed surface, σ, due to any distribution of charge, q, within the closed surface, is equal to the charge, q, multiplied by 4π*

We shall prove this theorem, first, for a point charge q_1, at a point M_1, within the surface, whose distance from any point, P, of this surface, is r. The potential at the point P, due to this charge, is

$$\phi = \frac{q_1}{r}$$

and the electric intensity

$$E = -\operatorname{grad} \phi = \frac{q_1}{r^2} \frac{r}{r}.$$

The flux of the intensity E through the surface σ will then be

$$\int_\sigma E \cdot n d\sigma = q_1 \int_\sigma \frac{\cos\theta d\sigma}{r^2} \tag{10.1}$$

where θ is the angle between r and n, the outer unit normal to the surface σ at the point P. If we surround the charge q_1 by a sphere of unit radius, center at M_1, then the cone, apex at M_1 and base $d\sigma$, will cut an area, $d\Omega$, out of the unit sphere, such that

$$\frac{d\Omega}{1} = \frac{d\sigma \cos\theta}{r^2}$$

(See Fig. 2.) The area $d\Omega$ so defined is called the *solid*

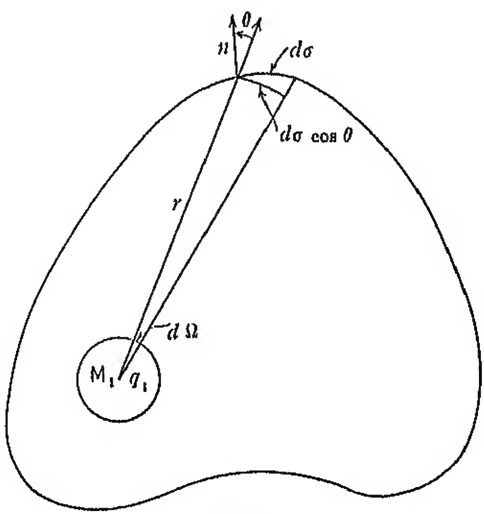


Fig. 2

angle of the cone. If we substitute this in equation (10.1), we shall have

$$\int_\sigma E \cdot n d\sigma = q_1 \int d\Omega = 4\pi q_1.$$

This result is independent of the position of the charge q_1 inside the surface Also, since the resultant intensity $\boldsymbol{E}$, due to two or more point charges, is the sum of the component intensities due to the separate charges, the flux of the intensity $\boldsymbol{E}$, through the surface σ, due to a system of n point charges entirely within the surface σ, will be

$$\int_\sigma \boldsymbol{E} \cdot \boldsymbol{n} d\sigma = 4\pi \sum_{i=1}^{n} q_i$$

For a continuous charge distribution of density ρ we replace q_i by $\rho_i d\tau$ and take the limit of the sum as we have done before, so that

$$\int_\sigma \boldsymbol{E} \cdot \boldsymbol{n} d\sigma = 4\pi \int_\tau \rho d\tau. \tag{10 2}$$

From this theorem we may easily deduce the corollary that

The flux of the intensity through the surface σ, due to a charge q_2, outside the surface σ, is zero.

For the flux of the intensity $\boldsymbol{E}_2$, due to the charge q_2 through a surface σ', surrounding the charge q_2 and the surface σ, is equal to $4\pi q_2$, according to the theorem just proved But if we regard q_2 as lying in the region bounded by the two surfaces σ and σ' then the flux of the intensity $\boldsymbol{E}_2$ through both surfaces is $4\pi q_2$. Hence, the flux through the surface σ of the intensity $\boldsymbol{E}_2$, due to the charge q_2, outside the surface, is zero We may thus state the general theorem that

The flux of the electric intensity $\boldsymbol{E}$, through a closed surface σ, where the intensity $\boldsymbol{E}$ is due to a system of charges embedded in the ether, is equal to the charge inside the surface σ, multiplied by 4π, or

$$\int_\sigma \boldsymbol{E} \cdot \boldsymbol{n} d\sigma = 4\pi \int_\tau \rho d\tau \tag{10.3}$$

where $\boldsymbol{E}$ is the intensity due to all charges, while the integral in the right member is to be taken over the charges interior to the surface σ, or throughout the volume bounded by σ.

Equation (10.3) is frequently referred to as Gauss's theorem; we shall call it *Gauss's electric flux* theorem to avoid confusing it with Gauss's flux divergence theorem.

If we transform the left member of equation (10.3) by Gauss's theorem (12), we obtain the equation

$$\int_\tau \operatorname{div} E d\tau = 4\pi \int_\tau \rho d\tau. \tag{10.4}$$

Since this equation holds for the element $d\tau$ we have

$$\rho = \frac{1}{4\pi} \operatorname{div} E$$

which expresses the volume density ρ, in terms of the intensity E

Had the electrification in τ been a surface distribution on the surface of a conductor of surface density ω, equation (10.3) would have read·

$$\int_\sigma E \quad n d\sigma = 4\pi \int_{\sigma_0} \omega d\sigma \tag{10.5}$$

where the first integral is to be taken over any arbitrary surface surrounding the charges, while the integral in the right member is to be taken over the charged surface. If we select for our surface, σ, one infinitesimally near to the charged conductor, then the two integrals in the limit will be extended over the same surface. To be consistent we should reverse the sense of the normal, n, taking the positive unit normal towards the interior of the conductor, thus making it the outward normal to the surface bounding the field of the intensity. With this change in the sense of the unit normal, equation (10.5) should read

$$-\int_\sigma E \cdot n d\sigma = 4\pi \int_\sigma \omega d\sigma.$$

The volume integral

$$\int_\tau \operatorname{div} E d\tau$$

in this case degenerates into a surface integral over the charged surface and is equal to the integral

$$4\pi \int_\sigma \omega d\sigma,$$

over the same surface.

Since Gauss's theorem holds whatever the volume, we may apply all this argument to an elementary tube of intensity of cross-sectional area $d\sigma$, and of infinitesimal length, and extend the tube into the conductor. Since the intensity $\boldsymbol{E}$ is tangent to the tube, and zero in the interior of the conductor, equation (10 5) in the limit as the length of the tube approaches zero, becomes

$$-\boldsymbol{E}\ \boldsymbol{n} d\sigma = 4\pi\omega d\sigma$$

or

$$\omega = -\frac{\boldsymbol{E}\ \boldsymbol{n}}{4\pi} = \frac{1}{4\pi}\boldsymbol{n} \text{ grad } \phi. \tag{10.6}$$

If we identify this with div $\boldsymbol{E}$, which in this case is also ω, we would term $-\boldsymbol{E}\ \boldsymbol{n}$ the *surface divergence* of $\boldsymbol{E}$, written briefly divs $\boldsymbol{E}$, so that $\frac{1}{4\pi}$ div $\boldsymbol{E}$ and $\frac{1}{4\pi}$ divs $\boldsymbol{E}$ are the volume and surface densities, respectively, of electrification In the language of Hydrodynamics these are the *sources* of the vector $\boldsymbol{E}$, and the vector lines of the vector $\boldsymbol{E}$ are *stream lines* We thus see that the lines of intensity begin at a positive charge and terminate on a negative * one in finite space or at infinity

When we take up the study of dielectrics we shall see that $-\boldsymbol{E}\ \boldsymbol{n}$ is only a special form for divs $\boldsymbol{E}$ In fact, *this* is more special than it appears, for the surface of a conductor is an equipotential surface, and since $\boldsymbol{E} = -$ grad ϕ, the intensity $\boldsymbol{E}$ is therefore normal to the surface, so that the surface density ω is $-\frac{|E|}{4\pi}$ or $+\frac{|E|}{4\pi}$,

* In hydrodynamic language the negative charge would be termed a sink, or a negative source, if the term source is used generically.

according as the sense of the intensity agrees with that of the unit normal n, or not

11. The Equations of Laplace and Poisson.—The intensity $E = -$ grad ϕ, thus the intensity is defined by the two equations

$$\frac{1}{4\pi} \operatorname{div} E = \rho, \quad \operatorname{rot} E = 0. \tag{11.1}$$

Since the second of these two equations is identically satisfied by the gradient of a scalar point function, the two equations (11.1) reduce immediately to the single equation

$$\operatorname{div} \operatorname{grad} \phi = -4\pi\rho$$

or

$$\Delta\phi = -4\pi\rho \tag{11.2}$$

This is Poisson's equation. For points where the density, ρ, is identically zero, the potential, ϕ, satisfies Laplace's equation

$$\Delta\phi = 0 \tag{11.3}$$

which is a special instance of Poisson's equation.

Equations (11.2) and (11.3) are very important; they arise everywhere in mathematical physics, in the conduction of heat, radiation, the theory of elasticity, hydrodynamics, and Newtonian attractions. The potential function used in the development of equation (11.2) is

$$\phi = \int \frac{\rho d\tau}{r}$$

already defined (6.3); it is thus a special solution of equation (11.2). Functions satisfying equation (11.3) are called harmonic functions.

Solving equations (11.1) for E is equivalent to solving equation (11.2) for ϕ. This is more convenient, since ϕ is a scalar point function. Then we can also prove that: *If a function ϕ satisfies Poisson's equation* (11.2) *over a region τ, and if ϕ takes on prescribed values at the surface σ, bounding the region τ, the function ϕ, is unique.* Also if $n \cdot$grad ϕ is given on the boundary σ, instead of the function ϕ, the

function ϕ is still unique aside from an additive constant In either case, the intensity E will be uniquely defined by the negative gradient of the function ϕ.

To prove this theorem we assume the existence of a second function ϕ' which satisfies equation (11 2) in the region τ, and which takes the prescribed values on the boundary We then replace the functions ϕ and ψ, of Green's theorem (18), which may be one and the same function, by $\phi' - \phi$.
Then

$$\int_\tau (\phi' - \phi)\Delta(\phi' - \phi)d\tau + \int_\tau \mathrm{grad}^2\,(\phi' - \phi)d\tau$$
$$= \int_\sigma (\phi' - \phi)\,\mathrm{grad}\,(\phi' - \phi)\;n d\sigma$$

But the first volume integral and the surface integral vanish identically, since

$$\Delta\phi' = \Delta\phi = -\,4\pi\rho$$

throughout the volume τ, and $\phi' = \phi$ on the surface σ. Therefore,

$$\int_\tau \mathrm{grad}^2\,(\phi' - \phi)d\tau \equiv 0$$

or

$$\mathrm{grad}^2\,(\phi' - \phi) = 0$$

If the square of the gradient of a scalar point function is zero, the point function is constant, or

$$\phi' - \phi = c$$

But $c = 0$ on the boundary, therefore $\phi = \phi'$ always, or in this case there is but one function, ϕ, satisfying the conditions

On the other hand, if $n\cdot\mathrm{grad}\,\phi$ is prescribed on the boundary, the volume and surface integrals will again vanish, the former for the same reason as before, while the surface integral becomes zero in this case because

$$n\cdot\mathrm{grad}\,\phi = n\cdot\mathrm{grad}\,\phi'$$

on σ. Thus again

$$\text{grad}^2 (\phi - \phi') = 0$$

or

$$\phi = \phi' + c$$

The conditions here determine ϕ aside from an additive constant; but this is no disadvantage, since in each case the intensity is uniquely determined as $-\text{grad}\ \phi$.

As a simple illustration, we shall determine the field intensity, $\boldsymbol{E}$, in free space, due to a surface charge, q, uniformly distributed over the surface of a spherical conductor. In this case the volume density, ρ, is everywhere zero, so that Laplace's equation

$$\Delta\phi = 0$$

is satisfied by the potential function, ϕ, at all points outside the surface of the conductor.

Since $\boldsymbol{E}$ is a Newtonian force intensity we seek a Newtonian potential function, one which vanishes at infinity and satisfies the condition:

$$\omega = \frac{1}{4\pi}\, n \text{ grad}\ \phi = \frac{q}{4\pi a^2}$$

at the boundary, the surface density being charged per unit area. We have already seen that

$$\text{grad}\, \frac{q}{r} = -\frac{q}{r^2}\,\frac{\boldsymbol{r}}{r}$$

if $\boldsymbol{r}$ is the radius vector drawn from the center of the sphere. Thus $\frac{\boldsymbol{r}}{r}$ and $\boldsymbol{n}$ are unit normals to the surface of the sphere, but they are of opposite sense, since $\boldsymbol{n}$ is the inward normal to the conductor. At the boundary, then, we will have

$$\frac{1}{4\pi}\left(n \text{ grad}\,\frac{q}{r}\right)_{r=a} = \frac{1}{4\pi}\left(\frac{q}{r^2}\right)_{r=a}$$

$$= \frac{q}{4\pi a^2}.$$

The function

$$\phi = \frac{q}{r}$$

vanishes at infinity, satisfies Laplace's equation at points outside the sphere and also the condition at the boundary, it is therefore the required function and the only one.

The field of the intensity

$$E = \frac{q}{r^2}\frac{r}{r}$$

is evidently a radial field, and we may add that the repulsive force acts at points outside the sphere as though the whole charge were concentrated at the center.

12. Dielectric Media and Condensers.—*Dielectrics* or *insulators* in the electron theory are characterized by the assumption that there are present in such media very few free electrons, nearly all electrons being strongly bound to the positive nuclei of the atoms The phenomena of induction is not exhibited by such non-conductors of electricity. There is no such thing in nature as a perfect conductor or a perfect insulator, but many media behave in such a way that they readily fall into one of the two classes

Maxwell's theory of dielectric media depends on Faraday's experiments with *condensers* A *condenser* consists of two oppositely charged conductors with an intervening dielectric, if the charged conductors are concentric spheres, the condenser is called a *spherical* condenser

The capacity of a condenser is defined as the quotient formed by dividing the charge on the positive plate by the difference of potential of the two plates. If S is capacity and $\phi_{21} \equiv \phi_2 - \phi_1$, where ϕ_2 is the potential at the positive plate, then by definition

$$S = \frac{q}{\phi_{21}}. \tag{12 1}$$

It is easily shown that the capacity of a spherical condenser in vacuo, i e., with the ether as the intervening dielectric, is constant, it is independent of the potential difference and the charge on the plate. For if q is the charge on the

sphere radius r_2, and if the radius of the other conductor $r_1 > r_2$, then

$$\phi_{21} = -\int_{r_1}^{r_2} E\,dr = -\int_{r_1}^{r_2} \frac{q\,dr}{r^2}$$

$$= \frac{q}{r_2} - \frac{q}{r_1} = \frac{q(r_1 - r_2)}{r_1 r_2}$$

so that

$$S = \frac{q}{\phi_{21}} = \frac{r_1 r_2}{r_1 - r_2}$$

which proves the statement. But Faraday found upon introducing a dielectric between the plates of a condenser, while maintaining the plates at a constant difference of potential, that the charge on the plates changed, or if we call S' the capacity and q' the new charge then

$$S' = \frac{q'}{\phi_{21}}.$$

In fact, his experiments showed that, *for every fixed difference of potential, whatever the shape of the condenser*, the ratio

$$\frac{S'}{S} = \frac{q'}{q}$$

remained constant for the same isotropic dielectric.

We shall call this ratio ϵ. This constant, ϵ, depending on the nature of the dielectric and capable of experimental determination, is called the *specific inductive capacity of the dielectric, or the dielectric coefficient.*

If we restrict ourselves to the spherical condenser, then we can show that if the charge on the plate remains constant, the difference of potential changes in a very special way. Initially in free space we shall assume that

$$S = \frac{q}{\phi_{21}}.$$

We then introduce an isotropic dielectric with dielectric coefficient, ϵ, keeping the charge, q, fixed while the differ-

ence of potential changes If we call the new difference of potential ϕ_{21}', and S'' the capacity, we shall have

$$S'' = \frac{q}{\phi_{21}'}$$

If we now maintain ϕ_{21}' fixed and remove the dielectric, the charge q will change to some value, q_0, while the capacity will assume its original value, S, for free space, since the capacity of a spherical condenser with the ether as the dielectric is independent of the charge on the plates and the difference of potential We thus have

$$S = \frac{q_0}{\phi_{21}'}$$

Hence from these last three equations, we find that

$$\frac{S''}{S} = \frac{q}{q_0} = \epsilon$$

and

$$\frac{S}{S} = 1 = \frac{q}{q_0}\frac{\phi_{21}'}{\phi_{21}}$$

so that

$$\phi_{21} = \epsilon\phi_{21}'$$

Since the new difference of potential is different from the old we conclude that the effect of introducing a dielectric between the condenser plates is a change in the electric intensity If we still use the letter E for the new intensity, then

$$\phi_{21} = \epsilon\phi_{21}' = -\int_{P_1}^{P_2} \epsilon E\, dr.$$

Thus the intensity in free space is seen to be the product of ϵ and the new intensity, so that the charge on the positive plate is now given by the integral

$$\frac{1}{4\pi}\int_\sigma \epsilon\, \mathbf{E}\cdot \mathbf{n} d\sigma$$

taken over a surface, σ, including the positive plate only on its interior. This new vector which has arisen is Max-

well's *electric displacement* vector, or simply the *displacement.*

We call this new vector D, and define it by the equation

$$D = \frac{1}{4\pi}\epsilon E. \tag{12.2}$$

It is characterized by the property that its flux through an arbitrary surface inclosing a charge *always* gives the charge interior to the surface Also if we are to maintain our definition for capacity, then the charge q must not only be given by the equation

$$q = \int_\sigma D \cdot n d\sigma \tag{12.3}$$

where the surface σ incloses the positive plate but also the new E must be a potential vector, i e, the negative gradient of a scalar point function

From (12.3) we have

$$q = \int_\sigma D \cdot n d\sigma = \frac{1}{4\pi}\int_\sigma \epsilon E \cdot n d\sigma.$$

But in free space according to Gauss's electric flux theorem

$$q = \frac{1}{4\pi}\int_\sigma E \cdot n d\sigma.$$

Hence in free space ϵ is unity and

$$D = \frac{1}{4\pi} E.$$

We thus have a natural unit dielectric coefficient in terms of which ϵ for every isotropic substance may be expressed. For such substances ϵ is the ratio of two charges; in general we shall regard it as a pure number of zero dimensions. The dielectric coefficient for paraffin is from 2 to 3; for glass, from 6 to 8, and nearly unity for every rarefied gas.

If the dielectric is non-homogeneous and non-isotropic we still define the displacement by equation (12.3) In this case the intensity E will be changed not only in mag-

nitude at every point of the medium, but in direction as well The dielectric coefficient here might be a *homography*, or the displacement D, a linear vector function of the intensity E, under the assumption that the relation (12.2) between the displacement and the intensity still holds. For our purpose, we need only remember that in general ϵ is a variable point function.

The relation (12 2) between the vectors D and E is interpreted in a mechanical way as a stress-strain relation, a special case of Hooke's law. In its most simple aspect, where the dielectric coefficient is constant, the vectors E and D would be interpreted as stress and simple extension, respectively, while $\frac{1}{\epsilon}$ would play the rôle of Young's modulus.

13. True and Fictitious Charges.—To discriminate more completely between the vectors E and D, we note first that by definition the flux of the displacement D, through every closed surface bounding a region τ, filled with any dielectric whatever, is equal to the charge in the region τ or

$$\int_\sigma D\ n d\sigma = \int_\tau \rho d\tau. \tag{13.1}$$

If we transform the left member of equation (13.1) by Gauss's theorem, this equation becomes

$$\int_\tau \operatorname{div} D d\tau = \int_\tau \rho d\tau$$

Since this is true for every volume, τ, it is true for the element $d\tau$, itself, so that the volume density of true electrification is always given by the equation

$$\rho = \operatorname{div} D.$$

If the electrification consists of true charges at an interface of two dielectrics (1) and (2), the integrals

$$\int_\tau \rho d\tau = \int_\tau \operatorname{div} D d\tau$$

degenerate into surface integrals over the interface The first integral becomes $\int_\sigma \omega d\sigma$, extended over the interface. If we transform the second integral by Gauss's theorem, we will have

$$\int_\tau \operatorname{div} D d\tau = \int_\sigma D\, n d\sigma$$

If we now contract the arbitrary surface σ, bounding the region τ, to coincide with the interface, we obtain the integral

$$\int_\sigma D\, n d\sigma$$

which must be extended twice over the interface, once over each side If D_1 and D_2 are the displacements and n_1 and n_2 the outer unit normals of the media (1) and (2), respectively, then the integral $\int_\sigma D \cdot n d\sigma$, taken over the bounding surface of medium (1), may be written $-\int_\sigma D_1\, n_1 d\sigma$, n and n_1 being just of opposite sense. The whole integral $\int_\tau \operatorname{div} D d\tau$ thus degenerates into

$$-\int_\sigma (D_1\, n_1 + D_2\, n_2) d\sigma.$$

If we now select an elementary displacement tube of cross-section $d\sigma$, for the region τ, and let the ends of the tube approach the interface, we shall have in the limit

$$\omega d\sigma = -(D_1 \cdot n_1 + D_2 \cdot n_2) d\sigma.$$

From this we conclude that the surface density

$$\omega = -(D_1\, n_1 + D_2\, n_2).$$

The divergence of the displacement D has thus degenerated into the expression

$$-(D_1 \cdot n_1 + D_2 \cdot n_2)$$

defined over the interface. This expression may be taken as the most general definition for the surface divergence

of a vector We may now say that the volume and surface densities of true electricity are given by the equations

$$\rho = \operatorname{div} D \text{ and } \omega = -(D_1 \cdot n_1 + D_2 \, n_2) = \operatorname{divs} D. \quad (13.2)$$

If there are no true charges in a dielectric medium or on the interface between two dielectrics

$$\operatorname{div} D = 0 \quad \text{and} \quad \operatorname{divs} D = 0 \qquad (13.3)$$

in these regions If the dielectric coefficient here is a variable scalar point function, equations (13.3) show that in these same regions

$$\operatorname{div} E \neq 0 \quad \text{and} \quad \operatorname{divs} E \neq 0.$$

Thus the intensity E has its sources not only in the true charges, but in those regions of a dielectric where ϵ is variable In these regions we assume "*fictitious electricity*" of volume, and surface densities ρ' and ω' defined by the equations

$$\rho' = \frac{1}{4\pi} \operatorname{div} E; \quad \omega' = \frac{1}{4\pi} \operatorname{divs} E. \qquad (13.4)$$

But the electric intensity is a potential vector defined by the equations

$$\frac{1}{4\pi} \operatorname{div} E = \rho'; \quad \operatorname{rot} E = 0 \qquad (13.5)$$

or, what is the same thing, the vector E is the negative gradient of a scalar point function, ϕ, so that equations (13 5) are equivalent to Poisson's equation

$$\Delta\phi = -4\pi\rho', \quad -(E \, n_1 + E_2 \, n_2) = 4\pi\omega'. \quad (13.6)$$

We thus see as before that the potential function ϕ, aside from an additive constant, and thus the intensity E, are uniquely determined from the sources of the vector E; these sources are in those regions where there are true and fictitious charges In fact, the function ϕ is a Newtonian potential function and may evidently be written in the form

$$\phi = \int_\tau \frac{\rho' d\tau}{r} + \int_\sigma \frac{\omega' d\sigma}{r}. \qquad (13.7)$$

14. The Vortices of the Displacement Vector.—When the curl of a vector is zero, the vector field in the language of Hydrodynamics is called an *irrotational* or *a vortex free* field The *vortices* of a *rotational* field are those regions where the curl of the vector is different from zero, and the curl of the vector is called the *strength* of the vortex. The field of the vector E is a vortex free field, since curl E is everywhere zero Therefore in a dielectric with a variable coefficient ϵ, curl D, according to relation (12.2), is different from zero Thus, in general, the displacement D is not an irrotational vector and therefore does not derive from a potential function

At an interface of two isotropic dielectrics with different dielectric constants, curl D is again different from zero, since curl E is zero In this case the vortices are distributed over a surface and are called *surface vortices*; curl D here degenerates into a *surface curl* which we will briefly write curls D.

If we apply Gauss's theorem to the vector curl D, taking for our volume an elementary displacement tube whose cross-section is a surface element, $d\sigma$, of the interface, we shall have

$$\operatorname{div} \operatorname{curl} D d\tau = \operatorname{curl} D \cdot n d\sigma.$$

The left member of this equation is identically zero since curl D is a solenoidal vector If we let the length of the tube approach zero, the surface element approaches an element of the interface as its limit, and since $d\sigma$ is arbitrary

$$\operatorname{curl} D \, n = 0$$

or curl D at a point of the interface lies in the plane tangent to the interface at the point.

To obtain the suggested special form for the curl of a vector we shall choose a right-handed set of unit vectors n_1, t, ν, and for convenience the coordinate axes OZ, OX, and OY, parallel, respectively, to these unit vectors.

We will choose n_1 as the outer unit normal to medium (1) at the interface, while t and ν are tangential to the interface If we now apply Stokes' theorem to the elementary rectangle $dzdx$ in the zx plane, with its sides dx in media (1) and (2), we will evidently get

$$\text{curl } D\ \nu dzdx = (D_2 - D_1)\ t dx + f dz$$

where fdz is written for the flow along the ends dz of the rectangle We now let the ends of the rectangle approach zero, the area $dzdx$ degenerates into the length dx, the term fdz becomes zero, while D_1 and D_2 approach their values at the interface in media (1) and (2). Thus in the limit we get

$$\text{curl } D\ n dx = (D_2 - D_1)\ t dx;$$

or since dx is arbitrary

$$\text{curl } D\ n = (D_2 - D_1)\ t$$

Had we taken the rectangle in the $n_1\nu$ plane, the unit vectors in our result would have been interchanged, and since interchanging ν and t gives us a left-handed set, we must write

$$\text{curl } D\ t = -(D_2 - D_1)\ \nu.$$

The perpendicular axes t and ν may be rotated, in their own plane about the normal n_1, and these two results will still persist The vectors curl D and $D_2 - D_1$ are therefore mutually perpendicular, they have the same magnitude, and they lie in the $t\nu$ plane. From these last two equations we must also conclude that n_1, $D_2 - D_1$ and curl D, in this order, form a right-handed set Hence

$$\text{curl } D = \text{curls } D = n_1 \times (D_2 - D_1)$$

where the cross ($\times$) is used to indicate the vector product. Or if we introduce the outer unit normal n_2 of medium (2), which is equal to $-n_1$, we shall have

$$\text{curls } D = -(n_1 \times D_1 + n_2 \times D_2).$$

This is the surface curl of the vector D, and may be taken as the form for the surface curl of every vector

At an interface of two isotropic dielectrics (1) and (2) with dielectric constants, ϵ_1 and ϵ_2,

$$\text{curl}_s\, E = -(n_1 \times E_1 + n_2 \times E_2) = 0;$$

or

$$n_1 \times E_1 = n_1 \times E_2$$

since curl E is everywhere zero. This last equation may also be written

$$|E_1| \sin (E_1, n_1) = |E_2| \sin (E_2, n_1) \tag{14 1}$$

Thus the tangential component of the electric intensity is continuous at an interface of two isotropic dielectrics. On the other hand, if there are no true charges at the interface, divs D is zero. We thus have from equation (13 2)

$$\text{div}_s\, D = -(D_1 \cdot n_1 + D_2\ n_2) = 0 \tag{14.2}$$

from which we see that

$$D_1 \cdot n_1 = D_2 \cdot n_1;$$

or

$$|D_1| \cos (D_1, n_1) = |D_2| \cos (D_2, n_1). \tag{14.3}$$

We may also say that if there are no true charges present, the normal component of the electric displacement is continuous at an interface. If we divide equation (14.1) by (14.3) and use relation (12.2) we easily obtain the relation

$$\frac{\text{Tan}\,(D_1, n_1)}{\text{Tan}\,(D_2, n_1)} = \frac{\epsilon_1}{\epsilon_2}$$

since D_1 and D_2 are parallel to E_1 and E_2, respectively. Thus the tangents of the angles which the lines of displacement make with the normal to the interface are directly proportional to the dielectric constants.

15. Space Distribution of Electric Energy.—We found (8.4) the mutual potential energy of the electric field in free space to be

$$W = \tfrac{1}{2}\int_\tau \phi\rho d\tau + \tfrac{1}{2}\int_\sigma \phi\omega d\sigma$$

which, since

$$-4\pi\rho = \Delta\phi \quad \text{and} \quad 4\pi\omega = \text{grad } \phi \cdot n$$

may be put into the form

$$W = -\frac{1}{8\pi}\int_\tau \phi\Delta\phi d\tau + \frac{1}{8\pi}\int_\sigma \phi \text{ grad } \phi \; n d\sigma.$$

If we transform the first integral by the first form of Green's theorem (18), the surface integrals will cancel, leaving

$$W = \frac{1}{8\pi}\int \text{grad}^2 \phi d\tau = \frac{1}{8\pi}\int E^2 d\tau. \qquad (15.1)$$

We thus see that the electric energy is distributed throughout all space with a volume density $\frac{1}{8\pi}E^2$.

If there are dielectrics in the field

$$\rho = \text{div } D \quad \text{and} \quad \omega = \text{divs } D = -D\cdot n$$

so that

$$W = \tfrac{1}{2}\int_\tau \phi \text{ div } D d\tau - \tfrac{1}{2}\int_\sigma \phi D\cdot n d\sigma$$

since, (17*a*),

$$\text{div } \phi D = \phi \text{ div } D + D \text{ grad } \phi$$

we may write the energy in the form

$$W = \tfrac{1}{2}\int_\tau \text{div } \phi D d\tau - \tfrac{1}{2}\int_\tau D \text{ grad } \phi d\tau - \tfrac{1}{2}\int_\sigma \phi D\cdot n d\sigma.$$

But by Gauss's theorem, the first and last integrals just cancel, so that

$$W = -\tfrac{1}{2}\int D \text{ grad } \phi d\tau = \tfrac{1}{2}\int D\cdot E d\tau. \qquad (15.2)$$

For space filled with an isotropic dielectric this reduces to

$$W = \frac{\epsilon}{8\pi}\int E^2 d\tau$$

In any case, then, we find that the electric energy may be

thought of as distributed throughout all space with a volume density $\frac{1}{2}D\ E$.

The integrals without the indices are to be taken throughout all space. The Green theorem is first applied to a finite region and then the boundary is extended to include all space. This is permissible since the integrals of (8.4) may include all space, those regions of space where ρ and ω are zero contribute nothing to the integrals.

16. Units and Dimensions.—To relate the rational electrostatic (r. e. s.) system of units to the common or absolute electrostatic (c. e. s.) system, we will call the unit charge in the (r. e. s.) system q_0 and in the (c. e. s.) system q_0'. Then by definition

$$\frac{q_0'^2}{1^2} = 1 \text{ dyne}$$

$$\frac{q_0^2}{1^2} = \frac{1}{4\pi} \text{ dyne}$$

the dielectric coefficient, ϵ, being unity in free space, so that

$$q_0'^2 = 4\pi q_0^2$$

or the (c. e. s.) unit charge is equal to the (r. e. s.) unit multiplied by $\sqrt{4\pi}$.

The natural unit charge or the charge carried by the electron has been determined by Millikan with great precision.* In (c. e. s.) units this was found to be $e = 4.774 \cdot 10^{-10}$ (c. e. s.) units, which is correct to one-half of one per cent. Expressed in the (r. e. s.) system units, this result would be multiplied by $\sqrt{4\pi}$.

The equation

$$F = qE \tag{16.1}$$

defines the electric intensity for every system of units. If the charge q is defined in (r. e. s.) units, then the intensity E is given in (r. e. s.) units intensity. Similar state-

* R. A. Millikan, "The Electron," p. 118.

ments may be made for the other quantities, the displacement D, and the potential ϕ

In order to obtain the connection between the electric intensity when measured in the two different systems of units, we shall denote quantities measured in the (c c s.) system of units by superscripts, then in this system

$$F' = q'E'.$$

If q and q' represent the number of (i c s) and (c c s) units, respectively, in a point charge situated in an electrostatic field, then

$$q = \sqrt{4\pi}q' \quad \text{and} \quad F' = F \tag{16 2}$$

or

$$q'E' = qE \tag{16 3}$$

If we substitute the first of equations (16.2) into equation (16.3) we find at once that

$$E' = \sqrt{4\pi}E$$

or the electric intensity expressed in (c c s.) units is equal to the electric intensity of the same field in (r. c s.) units multiplied by $\sqrt{4\pi}$ Also, from the definitions of the displacement and the potential, the relations

$$D' = \sqrt{4\pi}D \quad \text{and} \quad \phi' = \sqrt{4\pi}\phi$$

will be evident.

We wish now to find the dimensions of the electric quantities we have introduced In mechanics we were able to express all quantities dimensionally in terms of mass M, length L, and time T, and it is possible to express the electric quantities in terms of these same fundamental dimensions, providing we regard the dielectric coefficient as a mere number.

Our definition for force as the product of mass and acceleration leads to the following dimensional equation.

$$[F] = [M^1L^1T^{-2}]$$

If we write the equation

$$F = \frac{q^2}{r^2},$$

dimensionally, we shall have

$$[q] = [r\sqrt{F}] = [M^{1/2} L^{3/2} T^{-1}]$$

also

$$[E] = \left[\frac{F}{q}\right] = [M^{1/2} L^{-1/2} T^{-1}]$$

$$[D] = \left[\frac{\epsilon E}{4\pi}\right] = [M^{1/2} L^{-1/2} T^{-1}]$$

and

$$[\phi] = [E\ dr] = [M^{1/2} L^{1/2} T^{-1}]$$

It seems hardly necessary to justify the (c. e s) system of units adopted here However, this is the system most used by physicists, and the avoidance of more factors involving 4π by using the (i e s) system seems very doubtful. Then the change from one system to another is a very simple matter, as we have already seen.

EXERCISES

(1) Show that the average value of $\sin^2 t$ is $\frac{1}{2}$ if taken for a complete period

(2) If a charge q is uniformly distributed along a line of length $2l$, what will be the force intensity acting at a distance R from the line on its perpendicular bisector.

(3) Compute the potential function for the charge of problem 2

(4) Give the proof that A in Priestley's law is unity, for (c. e s) units and find A for the (i e s) system of units

(5) Prove that grad ϕ is a convergent integral if the charge is distributed in finite space, ϕ is the potential function,

$$\int \frac{\rho d\tau}{r}.$$

(6) Give a direct proof that

$$\Delta\phi = -4\pi\rho$$

where ϕ again is the electrostatic potential

(7) Find the potential energy of a body falling near the earth's surface

(*a*) At any instant, (*b*) At the instant it is 300 feet from the earth

(8) Find the potential energy due to the Newtonian attraction of a 1 pound mass, if this mass is 12,000 miles from the center of the earth

(9) An electric charge q is located at the center of a sphere, radius a, find the flux of the electric intensity E, due to this charge, through the surface of the sphere, by a direct integration.

(10) In the suggested experiment, Art 9, if the suspended insulated body is brought into contact with the inner surface of the hollow conductor, what will be the resulting phenomenon? Describe the fields inside and outside the hollow conductor.

(11) Give a different proof that a sphere radius, a, carrying a uniform positive charge q, will repel a unit positive charge at a distance $r > a$ from the center of the sphere as if its mass were concentrated at the center of the sphere.

CHAPTER II

MAGNETISM AND POLARIZED DIELECTRICS

1. **Magnetism** and certain magnetic phenomena are matters of common knowledge It was discovered very early that loadstone, a particular kind of iron ore, possesses magnetic properties This inherent magnetism, or *intrinsic magnetization*, is characteristic of permanent magnets That the earth is a magnet has been known for a long time, though just what is the source of the earth's magnetism is a question which has never been conclusively answered.

Surrounding a magnet there is a magnetic field of force intensity. If a bar magnet be suspended so that it swings freely in a horizontal plane, it will line itself up in a north and south direction Near the ends of the magnet and not very well defined are two points called the *magnetic poles*, from which the force seems to arise The line joining these two points is called the *axis* of the magnet. The pole of the magnet which turns to the north is usually called the *positive pole* and the other the *negative pole*

There is a formal resemblance between magnetism and electricity, though a magnetic pole has no influence at all on a static charge of electricity. Like poles repel and unlike poles attract each other, as did like and unlike electric charges Also, by a series of extensive experiments, Gauss established the fact that, for concentrated poles, the Newtonian law of force holds

As we defined the unit charge in electrostatics, so we define a *unit magnetic pole as a pole which repels a like pole in vacuo at a distance of one centimeter with a force of one dyne.* The unit pole here defined is the common electromagnetic unit pole; it is the basis for the common electro-

magnetic (c e m) system of units. A unit pole is also said to be of unit *strength*

If m_1 and m_2 are the strengths of two like poles, and if r is their distance apart, the repulsive force, F, may then be written in the form

$$F = A\frac{m_1 m_2}{r^2}\frac{\mathbf{r}}{r} \tag{1.1}$$

which is the Newtonian law of force. The factor A is again a universal constant depending on the choice of units If we apply the (c e m.) system of units to equation (1 1), taking r equal to one centimeter, F equal to one dyne, and m_1 and m_2 each equal to one positive (c e m.) unit pole, we find that the constant A is unity, so that

$$F = \frac{m_1 m_2}{r^2}\frac{\mathbf{r}}{r} \tag{1.2}$$

in our chosen system of units

The *magnetic field intensity*, H, suitably defined by the equation $F = mH$, is seen to be a force intensity or force per unit pole strength Thus H, like the electric field intensity E, and similarly defined, is a vector point function such that if a pole of pole strength m be brought into the field without changing the field, mH will be the force at the point acting on the pole Also if magnetic material be brought into the field it will be found to be magnetized by induction We shall therefore speak of *induced magnetism* as well as induced electricity.

2 The Elementary Magnet.—We now wish to characterize the magnetic field due to a continuous distribution of magnetism The dissimilarity between the fields of magnetic and electric intensity and between their potentials is due to the fact that it is impossible to isolate a magnetic pole We thus have to deal with a system of bi-poles or point doublets. If we break a magnet up into pieces as small as possible, the pieces persist in the same magnetic properties possessed by the original magnet. A magnet

is thus thought of as a polarized medium, a medium composed of such point doublets. The nearest approach to an isolated pole is one pole of a very long magnet of small cross-section. In the region surrounding one end of such a magnet the effects of the other pole would be negligible.

The whole theory of magnetism is based on the idea of the *elementary magnet* To develop this idea we write out the Newtonian potential for a very small magnet and neglect second order terms, which drop out in the limiting process of the integral calculus. The negative gradient of this function will give the magnetic intensity H correct to second order infinitesimals.

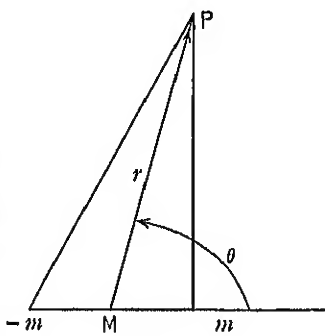


FIG. 3.

The Newtonian potential, ψ', at a point, P, of a very small magnet of length h, and pole strength m, in vacuo, whose center is at the point M, distance r from P, is evidently (Fig. 3)

$$\psi' = \frac{m}{r - \frac{h}{2}\cos\theta} - \frac{m}{r + \frac{h}{2}\cos\theta} = \frac{mh\cos\theta}{r^2 - \frac{h^2}{4}\cos^2\theta}$$

where θ is the angle between the vector r, drawn from the point M to the point P, and the vector dM, drawn from $-m$ to m, whose modulus is h. (We adopt the Piano notation and write

$$r = P - M$$

for the vector drawn from M to P. Then

$$dr = dP = -\,dM.\text{*})$$

We now regard h as an infinitesimal and neglect all second

* Since dr is an arbitrary displacement of an end point of r, this notation discriminates explicitly between the displacements of M and P.

order terms, and define the potential, ψ, of the elementary magnet by the equation

$$\psi = \frac{mh\cos\theta}{r^2}.$$

This will be the potential of the original magnet correct to second order terms

The potential of the elementary magnet may be put into a more convenient form by first observing that

$$dr = \operatorname{grad} r\ dr = \operatorname{grad}_P r\ dP$$
$$= \operatorname{grad}_M r\ dM = -\operatorname{grad}_M r\ dP$$

or

$$\operatorname{grad}_P r = -\operatorname{grad}_M r$$

Also since $\operatorname{grad}_P r$ is a unit vector parallel to $\mathbf{r} = P - M$, and $h = |dM|$,

$$h\cos\theta = |dM|\ |\operatorname{grad}_P r|\cos\theta$$
$$= \operatorname{grad}_P r\ dM = -\operatorname{grad}_M r\cdot dM$$

Thus

$$\psi = \frac{mh\cos\theta}{r^2} = -\frac{mdM\ \operatorname{grad}_M r}{r^2}$$
$$= mdM\ \operatorname{grad}_M \frac{1}{r}$$

where $\frac{-1}{r^2}\operatorname{grad}_M r$ has been replaced by its equal, $\operatorname{grad}_M \frac{1}{r}$.

The vector mdM is called the *moment* of the magnet; if we write $\mathbf{m}$ for this moment, the potential of the elementary magnet will take the final form

$$\psi = \mathbf{m}\ \operatorname{grad}_M \frac{1}{r} \tag{2.1}$$

It should be strongly emphasized that we are not here interested in the fundamental nature of magnetism; we are interested in a consistent theory that takes care of the average effects So in the study of magnetic media we are still interested in average values We are not attempting to dissect and analyze the atom or molecule; in fact,

we regard the elementary magnet as made up of many molecules

3. Intensity of Magnetization—Since a magnetic pole can not be isolated, the moment of the magnet rather than its pole strength plays the more important rôle. For if an elementary magnet be placed in a magnetic field of intensity H, it will be acted on by a torque or couple We will define the positive sense of the magnetic intensity $\boldsymbol{H}$ as the sense in which a positive pole will be driven. We may then say that the couple tends to rotate the magnet into such a position that its own lines of intensity have the direction and sense of the field intensity $\boldsymbol{H}$

The moment of a couple is usually interpreted as a vector, perpendicular to the plane of the couple, whose magnitude is the product of the magnitude of the force and the arm of the couple. This magnitude will be the area of the parallelogram whose altitude is the arm of the couple and one of whose sides is the force $\boldsymbol{H}$ If the moment of the elementary magnet is m, then the moment of the couple will evidently be given by the vector $\boldsymbol{m} \times \boldsymbol{H}$ The positive sense of the moment of the couple is so chosen that the magnet will appear to rotate in a counter-clockwise sense, as seen from the terminal of the moment of the couple.

In a volume element $d\tau$ of a volume distribution of magnetism, the resultant moment of all elementary magnets in the element may be effectively replaced by a single vector. We are thus led to introduce a new vector, I, the moment per unit volume, called the *intensity of magnetization*, and such that $Id\tau$ will give the resultant moment of all the elementary magnets in the element $d\tau$ The intensity of magnetization, I, may be a vector point function; in such a case the polarization is of the most general type We thus see that if a magnet be placed in a magnetic field of intensity $\boldsymbol{H}$, every volume element will be acted upon by a couple the moment of which will now be given by the vector $Id\tau \times \boldsymbol{H}$.

4. The Scalar Magnetic Potential.—The potential of an elementary magnet situated at a point M was given by equation (2.1) to be the scalar product, $m \operatorname{grad}_M \frac{1}{r}$ To obtain the magnetic potential for an element $d\tau$ of a *permanent* magnet, we would sum the potentials of all elementary magnets in the volume element This will be equivalent to replacing the moment of the elementary magnet by the resultant moment of all the elementary magnets in the element $d\tau$. But this resultant is given in terms of the intensity of magnetization by the vector $Id\tau$; hence the magnetic potential at a point P, exterior to the magnet, for the element $d\tau$, situated at a point M of the magnet is $I \operatorname{grad}_M \frac{1}{r} d\tau$.

If induced magnetism is in the field also, we shall have other polarized regions, with a resultant magnetic moment in each volume element and therefore a moment per unit volume or an intensity of induced magnetization Therefore, the scalar magnetic potential for any number of permanent and temporary magnets at a point P, outside the magnetic media, will be given by the integral

$$\psi = \int_\tau I \operatorname{grad}_M \frac{1}{r} d\tau \tag{4 1}$$

taken throughout all magnetic media.

We might observe again that while equation (2.1) gives only the approximate potential for a small magnet, equation (4 1) gives exactly the potential for a space distribution of magnetic material We have only dropped out in the procedure the second order terms which automatically drop out in applying the fundamental law of the integral calculus

Equation (4 1) defines the magnetic intensity at every point, P, outside the magnetic media. But for points within the media the integrand of the potential function becomes infinite to the second degree when M coincides

with the point P, since grad $\frac{1}{r}$ involves the factor r^{-2}. If we apply the transformation theorem (17a),

$$\operatorname{div} \phi u = \phi \operatorname{div} u + u \operatorname{grad} \phi$$

to the integrand, the potential function (4.1) will take the form

$$\psi = -\int_\tau \frac{\operatorname{div} I d\tau}{r} + \int_\tau \operatorname{div} \frac{I}{r} d\tau$$

If we now transform the second of these two integrals by Gauss's theorem we get

$$\psi = -\int_\tau \frac{\operatorname{div} I d\tau}{r} + \int_\sigma \frac{I\, n d\sigma}{r} \qquad (4.2)$$

The first integral is to be taken throughout the volume of all magnetic media, and the second integral over the bounding surfaces of these media, n being the outer unit normal.

The magnetic potential ψ, grad ψ, and $\Delta\psi$ are uniquely defined by equations (4.1) or (4.2) for points outside the magnetic media. But for points of the magnetic media, $\Delta\psi$ is no longer defined through equation (4.1), for $\Delta\psi$ will then involve r^{-4} as a factor of the integrand. On the other hand, ψ, grad ψ, and $\Delta\psi$ are uniquely defined at all points of space by equation (4.2), since here ψ is a Newtonian potential function. We thus analytically extend the meaning of ψ for points of the magnetic media by taking equation (4.2) as its general definition. The magnetic potential thus becomes a Newtonian potential function defined everywhere. The potential ψ, whose negative gradient is the magnetic intensity H, is thus the Newtonian potential for a mass of volume density —div I and a surface distribution of surface density $I \cdot n$.

5. The Equations of Poisson and Laplace.—We have already seen that the Newtonian potential function

$$\phi = \int_\tau \frac{\rho d\tau}{r}$$

satisfies Poisson's equation

$$\Delta\phi = -4\pi\rho$$

therefore Poisson's equation foi the magnetic potential will take the form

$$\Delta\psi = 4\pi \operatorname{div} I \tag{5.1}$$

At the boundary of the magnetic media the right membei degenerates into

$$4\pi \operatorname{divs} I = -4\pi(n_1 \; I_1 + n_2 \; I_2);$$

the intensity of magnetization, I_1 oi I_2, will be zeio if the magnetic material lies adjacent to a non-magnetic media or free space. Also in free space div I is zero, so that

$$\Delta\psi = 0$$

or Laplace's equation is satisfied at points outside the magnets Then since

$$-\operatorname{grad}\psi = H$$

Poisson's equation may also be written in the form

$$\Delta\psi = -\operatorname{div} H \tag{5 2}$$

If we compare this equation with (5 1) we see that

$$\operatorname{div}(H + 4\pi I) = 0 \tag{5.3}$$

and at a boundaiy of the magnetic media then

$$\operatorname{divs}(H + 4\pi I) = 0 \tag{5 4}$$

This new vector, $H + 4\pi I$, which has incidentally aiisen, is destined to play a fundamental rôle in the theory of magnetism

6. The Magnetic Induction.—The new vectoi, $H + 4\pi I$, which appeared in the previous section, is of fundamental importance This vector, which we shall designate by the letter B, is called the *magnetic induction* or simply the *induction*, and corresponds to the electric displacement D. The induction B, defined in (c. e. m) units by the equation

$$B = H + 4\pi I \tag{6.1}$$

is a source free or solenoidal vector, since according to equation (5 3) its divergence is everywhere zero But the divergence of the curl of every vector is also identically zero, we may therefore look upon the induction as the curl of another vector A and write

$$B = \text{curl } A \tag{6 2}$$

The vector A from which the induction B may be derived is Maxwell's vector potential

If we care to draw an analogy with electrostatics, where the true electric density was given by div D and divs D, we may write

$$\rho_m = \text{div } B, \quad \omega_m = \text{divs } B \tag{6.3}$$

for the volume and surface densities of true magnetism But since div B and divs B are identically zero, ρ_m and ω_m are zero everywhere We should expect this, since neither positive nor negative magnetism ever exists alone in nature In every volume element of a magnetized medium they exist in equal amounts, so that the density of true magnetism is everywhere zero

If we write out Gauss's theorem for the induction

$$\int_\tau \text{div } B d\tau = \int_\sigma B \cdot n d\sigma$$

we see that since the induction B is a solenoidal vector, div B vanishes, and we have the result that the total flux of induction through every closed surface is zero. Hence the vector lines of the vector B, the lines of induction, are closed curves, and the tubes of induction are therefore closed tubes. Some of the lines of induction may be closed at infinity, but none of them terminate in finite space, for if they did the flux of induction through a closed surface including these terminals would evidently not be zero, which would contradict the fact that the induction is a solenoidal vector.

7. Fictitious Magnetism—The magnetic intensity H, like the electric intensity E, is determined from its sources,

since it is a lamellar or potential vector The sources of the vector H are found where div H is different from zero Since div B is everywhere zero, and the induction B is by definition $H + 4\pi I$, we have

$$\frac{1}{4\pi}\operatorname{div} H = -\operatorname{div} I; \quad \frac{1}{4\pi}\operatorname{divs} H = -\operatorname{divs} I. \tag{7.1}$$

We thus see that the sources of the magnetic intensity are situated not only in those regions occupied by permanent and temporary magnetism of variable intensity of magnetization, I, but also on the boundaries of these regions and at an interface of two different magnetic media with different intensities of magnetization In these regions we thus assume *fictitious magnetism* of volume and surface densities.

$$\rho'_m = \frac{1}{4\pi}\operatorname{div} H, \quad \omega'_m = \frac{1}{4\pi}\operatorname{divs} H \tag{7.2}$$

respectively. If we substitute the first of (7.1) into (5.1) we see that Poisson's equation becomes

$$\Delta\psi = -4\pi\rho'_m.$$

The conditions at the boundary are given by the equations

$$\operatorname{divs} B = 0; \quad \operatorname{curls} H = 0 \tag{7.3}$$

the former leading to the second of (7.1) which may be written more explicitly in the form

$$-\frac{1}{4\pi}(n_1 \cdot H_1 + n_2 \cdot H_2) = (n_1 \cdot I_1 + n_2 \cdot I_2)$$

and the latter to

$$\operatorname{curls} H = -(n_1 \times H_1 + n_2 \times H_2) = 0$$

at an interface between two magnetic media (1) and (2). If we replace H_1 by $-\operatorname{grad}\psi_1$, and H_2 by $-\operatorname{grad}\psi_2$, these two equations become

$$\left.\begin{aligned} \frac{1}{4\pi}[n_1 \cdot \operatorname{grad}\psi_1 + n_2 \cdot \operatorname{grad}\psi_2] &= n_1 \cdot I_1 + n_2 \cdot I_2 \\ n_1 \times \operatorname{grad}\psi_1 + n_2 \times \operatorname{grad}\psi_2 &= 0 \end{aligned}\right\} \tag{7.4}$$

In case one of the media, possibly (2), is non-magnetic, I_2 will vanish, giving a special form for the first of (7 4). These boundary conditions (7 4) are expressed in the statement that the normal component of the induction B, and the tangential component of the magnetic intensity H, are continuous at an interface of two magnetic media This situation is analogous to that holding for the electric displacement D, and the electric intensity E, at an interface of two dielectrics

The explicit form for the potential ψ in terms of the volume and surface densities of fictitious magnetism is easily found to be

$$\psi = \frac{1}{4\pi}\int_\tau \frac{\rho'_m d\tau}{r} + \frac{1}{4\pi}\int_\sigma \frac{\omega'_m \sigma}{r}$$

by substituting (7 1) and (7 2) into (4 2) The first integral is to be taken throughout the volume of all magnetic media, while the latter is to be extended over the boundaries of these media. Again the potential ψ, just expressed, is a special solution of Poisson's equation.

8. The Potential of a Magnetic Shell—We define a magnetic shell as a surface distribution of elementary magnets with a common normal sense; each magnet has its center on this surface and its axis perpendicular to it. If the thickness of the shell is dn and the intensity of magnetization I, then the modulus of the vector Idn, which we will designate by Φ, is called the *strength of the shell.* The vector Idn is evidently the moment per unit area

The potential of the shell at a point P, outside the shell, given by equation (4.1), may be written in the form

$$\psi = \int_\sigma Idn \text{ grad} \frac{1}{r} d\sigma = \int_\sigma \frac{\Phi d\sigma \cos\theta}{r^2} \tag{8.1}$$

where θ is the angle (I, r), the vector I being normal to the surface of the shell at some point, M, while r is the vector drawn from M to P. The form of the potential

given here is the general form for the *double layer potential* In the study of this function, the strength of the shell, Φ, plays the rôle of a density factor differing from zero at points of the surface We arrive at this idea in a more exact way by taking the limit of $|Idn|$, as dn approaches zero, under the assumption that the intensity of magnetization I increases without limit in such a way that the product Idn remains finite This limiting value of the modulus of Idn is taken as the density factor, Φ.

If we cut the surface σ by a secant plane perpendicular to the normal to the surface at P and near to P, this plane will slice off a surface cap which we will designate as C_1 We shall call the distance from the secant plane to the point P, ϵ. Then as ϵ approaches zero the secant plane will approach as its limiting position the tangent plane at the point P, providing the orientation of the secant plane remains unchanged Now the magnetic shell has natural positive and negative sides, determined by the sense of the elementary magnets Also the cap C_1 together with that portion of the secant plane lying outside the boundary of C_1 divide all space into two parts We shall call that portion of space bordering on the positive face of the cap the positive side of the cap, and the other part the negative side. Also the solid angle subtended at a point P', by the boundary of the cap C_1, will be defined as positive or negative according as P' lies on the positive or negative side of the cap

We shall choose the point P' on the positive side of the cap C_1, and at a distance r from an element $d\sigma$ of the cap We shall now draw a unit sphere and a sphere radius r, with P' as center. Then the elementary cone whose apex is P' and whose base is $d\sigma$ cuts out a solid angle on the unit sphere which we shall call $d\Omega$, and the area $d\sigma \cos \theta$ on the sphere radius r, if θ is the angle between the normal to the element $d\sigma$ and the vector drawn from the point M of the element $d\sigma$ to the point P'. Since these areas are proportional to the square of their radii (Fig. 2), we shall have

$$\frac{d\Omega}{1^2} = \frac{d\sigma \cos \theta}{r^2}$$

and

$$\Phi d\Omega = \frac{\Phi d\sigma \cos \theta}{r^2}$$

where Φ is the strength of the shell at the point of the cap Thus the potential of the cap at the point P' will be given by the integrals

$$\psi_1 = \int_{C_1} \frac{\Phi \cos \theta d\sigma}{r^2} = \int \Phi d\Omega$$

where the second integral is to be taken over that portion of the unit sphere lying inside the cone whose apex is P' and whose base is the boundary of the cap C_1 If we call this solid angle Ω' and if Φ is constant for points of the cap, then

$$\psi_1 = \Phi \ \Omega'$$

But in the case under consideration the strength Φ of the elementary shell varies over C_1 so that if Φ' and Φ'' are the minimum and maximum values of Φ on C_1, then the potential at P' will be the product of the solid angle Ω and some value of Φ between Φ' and Φ'' We shall call this value Φ_1 so that $\Phi_1 \Omega'$ is then the potential of C_1 at P'.

If we let P' approach P, the solid angle Ω' will approach as its limit the solid angle at P subtended by the boundary of the cap. If we call this solid angle Ω, then the potential at P due to the cap will be equal to $\Phi_1 \Omega$

If we write σ_1 for $\sigma - C_1$, then the potential at the point P of the shell is

$$\psi = \int_{\sigma_1} \frac{\Phi \cos \theta d\sigma}{r^2} + \Phi_1 \Omega \tag{8.2}$$

Since the integral over σ_1 is finite and $\Phi_1 \Omega$ is finite, we conclude that the potential

$$\psi = \int_{\sigma} \frac{\Phi \cos \theta d\sigma}{r^2}$$

at a point P' outside the shell converges, i.e., approaches a finite value, as P' approaches the point P of the shell But as ϵ approaches zero or as the secant plane approaches the tangent plane at P, the solid angle Ω will approach the value 2π if P' is on the positive side of C_1, and Φ_1 will approach the value Φ at the point P while the integral over σ_1 has as its limit the integral over σ. In the limit then we find that

$$\psi = \int_\sigma \frac{\Phi \cos \theta d\sigma}{r^2} + 2\pi\Phi$$

is the value of the double layer potential at a point P of the shell when P is approached from the positive side of the cap But we have already proved (8.2) that in this case ψ is finite, hence the integral in the above equation is a convergent integral.

Had the direction of approach been towards the negative face of the shell, the additive term would have been $-2\pi\Phi$ If we discriminate between these two values of ψ, calling the former ψ_e and the latter ψ_i, we may write

$$\left.\begin{aligned} \psi_e &= \int_\sigma \frac{\Phi \cos \theta d\sigma}{r^2} + 2\pi\Phi \\ \psi_i &= \int_\sigma \frac{\Phi \cos \theta d\sigma}{r^2} - 2\pi\Phi. \end{aligned}\right\} \tag{8.3}$$

If we combine these two equations, we shall get the equations

$$\left.\begin{aligned} \psi_e - \psi_i &= 4\pi\Phi \\ \psi_e + \psi_i &= 2\int_\sigma \frac{\Phi \cos \theta d\sigma}{r^2}. \end{aligned}\right\} \tag{8.4}$$

These results are extremely important, the first equation in the study of current electricity, while both equations are used in the modern proof of Dirichlet's principle, which, of course, is out of this province

The double layer potential

$$\psi = \int_\sigma \frac{\Phi \cos \theta d\sigma}{r^2}$$

is a unique single valued function of the point P, at every point of space; this fact is expressed for points of the shell by equations (8 3) The value of the function at a point of the shell is the value of the integral taken over the shell when P is a point of the shell; we have proved that this is a convergent integral But we have seen that the function approaches the shell discontinuously, it does not converge to the value of the function at the point of the shell approached, but to this value, $+2\pi\Phi$ or $-2\pi\Phi$, according as the approach is towards the positive or negative face, respectively.

Just as the electrostatic potential had a natural zero value at infinity, so also has the scalar magnetic potential Thus, since

$$d\psi = \text{grad}\,\psi\, dr$$

and

$$H = -\,\text{grad}\,\psi$$

we may write the double layer potential as the line integral

$$\psi = -\int_{\infty}^{P} H \cdot dr \tag{8.5}$$

where it is assumed that ψ vanishes at infinity This assumption only fixes the additive constant, whose value is immaterial. The path of integration from infinity to the point P is arbitrary since the integral depends only on the end points of the path. This function has a unique value at a point P_2, taken near the positive face of the shell. If we extend the path of integration from P_2 around the edge of the shell back through the shell to P_2 the function will assume its original value at P_2, or the *circulation integral,* $\int_s H\, dr$, *from* P_2 *around through the shell in positive sense back to* P_2, *is zero* Also, according to the first of (8.4), $\psi_o = \psi_i + 4\pi\Phi$, so that (and this is another important result for current electricity) the circulation integral (or the magnetic potential) *increases by* $4\pi\Phi$ *in passing through the magnetic shell in positive sense.*

It may be well to state explicitly again the conditions under which these important results have been obtained. For points of the magnetic media we have seen that ψ must be defined by the equation (4.2),

$$\psi = -\int_\tau \frac{\operatorname{div} I d\tau}{r} + \int_\sigma \frac{I\ n d\sigma}{r}$$

in order that $H = -\operatorname{grad} \psi$ and $\Delta\psi$ have a meaning at such points. Accordingly, at points of a volume distribution of magnetism, we found that

$$\Delta\psi = 4\pi \operatorname{div} I = -\operatorname{div} H$$

while at the boundary of such media this result degenerated into a surface divergence or

$$-\operatorname{divs} \operatorname{grad} \psi = + \operatorname{divs} H = -4\pi \operatorname{divs} I$$

and more explicitly

$$-(n_1\ H_1 + n_2\ H_2) = 4\pi I\ n_2.$$

This shows just how the intensity H behaves at a point of the double layer, for the last equation may be written in the form

$$-(H_2 - H_1)\ ndn = 4\pi Idn \cdot n$$

where $n = n_2 = -n_1$. Since $|Idn| = \Phi$, the right member of this last form is $4\pi\Phi$, while the left member is that element of the circulation integral $\int_0 H \cdot dr$ at the point P of the double layer. Our results hold at every point of a complex shell, Φ, the strength of the shell, being a variable point function, the simple shell, Φ constant, is a special case which needs no separate treatment.

9. **The Energy of the Magnetic Field.**—In the study of the potential energy of the electric field we obtained the fundamental result (I, 7.4) that, if the potential function vanished at infinity, the potential energy of a charge q, at a point P, in an electric field, was equal to the product of the charge and the potential at the point P. Since the

magnetic potential is a Newtonian potential function, we can apply this result directly to an elementary magnet whose negative pole of pole strength m is at a point P. If the potential at the point P, due to certain magnetic media, is ψ, then the potential at the positive pole will be $\psi + d\psi$, so that the potential energy of the magnetic doublet is

$$-m\psi + m(\psi + d\psi) = md\psi$$
$$= m d\mathbf{r} \text{ grad } \psi = \mathbf{m} \text{ grad } \psi$$

where we have written the vector $\mathbf{m}$ for the moment $md\mathbf{r}$ of the doublet as formerly.

Also the mutual potential energy of a system of n electric charges was given (I, 8 3) by the expression

$$\sum_{i=1}^{n} \tfrac{1}{2} q_i \phi_i$$

the sum of one-half the potential energy of each charge. This result is directly applicable to a system of n elementary magnets, or one may go through the longer process of bringing the doublets up from infinity in order. In either case, we arrive at the result that the mutual potential energy of the system of magnetic doublets is

$$\tfrac{1}{2} \sum_{i=1}^{n} \mathbf{m}_i \text{ grad } \psi_i,$$

where ψ_i is the potential function at the point P_i of all the doublets except the doublet whose moment is $\mathbf{m}_i$. If there is no induced magnetism in the field, the potential energy W_m, for a continuous distribution of *permanent magnetism* of intensity of magnetization $\mathbf{I}$, thus becomes

$$W_m = \tfrac{1}{2} \int_\tau \mathbf{I} \text{ grad } \psi d\tau = -\tfrac{1}{2} \int_\tau \mathbf{I} \cdot \mathbf{H} d\tau \qquad (9.1)$$

where the integral may be taken throughout all permanent magnets, or throughout all space, since the vector $\mathbf{I}$ is zero outside the magnetic material.

If we replace the intensity of magnetization I, in equation (9 1), by its equal $\frac{1}{4\pi}(B - H)$, the potential energy will assume the form

$$W_m = -\frac{1}{8\pi}\int (B - H)\ Hd\tau$$
$$= -\frac{1}{8\pi}\int B\cdot Hd\tau + \frac{1}{8\pi}\int H^2 d\tau$$

where the integration is extended throughout all space. It is easy to prove that in a magnetic field due to magnetic media only, the first integral, $\int B\ Hd\tau$, vanishes when taken throughout all space. This may be seen by first replacing the intensity H by $-\text{grad}\,\psi$, and then, transforming the resulting integrand, B grad ψ, by formula (17a)

$$\text{div}\,\psi B = \psi\,\text{div}\,B + B\cdot\text{grad}\,\psi;$$

we find that the integral

$$\int B\cdot Hd\tau = -\int \text{div}\,\psi Bd\tau + \int \psi\,\text{div}\,Bd\tau.$$

Since div B is identically zero, the second integral in the right member of this equation vanishes, and if we surround all magnetic media by a spherical surface σ, we may transform the first integral by Gauss's theorem into the surface integral $\int_\sigma \psi B\cdot n d\sigma$. But the induction

$$B = H = -\text{grad}\,\psi$$

in free space, so that we may replace the induction B by $-\text{grad}\,\psi$ in this surface integral; we will thus have finally

$$\int_\tau B\ Hd\tau = \int_\sigma \psi\,\text{grad}\,\psi\cdot n d\sigma.$$

The potential ψ vanishes like r^{-2} at infinity, and its negative gradient like r^{-3}, while the spherical surface becomes infinite like r^2, we may conclude that this surface integral

vanishes in the limit as the radius of the sphere increases without limit, and that the volume integral, $\int B\,Hd\tau$, extended throughout all space, is zero *

The potential energy may thus be written in the simple form

$$W_m = \frac{1}{8\pi}\int H^2 d\tau$$

where the integral is to be taken throughout all space. This is Maxwell's result †

This result of Maxwell, though correct, must not be wrongly interpreted; if $\frac{1}{8\pi} H^2$ is to be interpreted as energy per unit volume it must be thought of only in an average sense. The possible error in interpreting Maxwell's result lies in the fact that the energy in the regions occupied by magnetic media is not put into evidence. If we can separate off this purely local part of the energy, we shall be in a position to define the energy density at every point of space.

In doing this we shall take the more general case where induced or temporary magnetism is in the field of the permanent magnets. The total energy will be the work done in bringing the permanent magnets up from a state of infinite dispersion, the induced or temporary magnetism adjusting itself at every step of the process. This energy will evidently be given by formula (9 1); in form this is

* For a more detailed proof we may replace r in the potential function and its gradient by its minimum value, r_0, we shall then have

$$|\psi| \leq \frac{1}{r_0}\left|\int \rho'_m d\tau\right| = \frac{m_0}{r_0}, \text{ etc}$$

† Some of Livens' criticisms are pertinent here Livens points out that the treatment of the subject of magnetic energy "in the usual text-books is hopelessly confused" This is due to reasoning by analogy and a lack of discrimination between potential and kinetic energy See G. H Livens "On the Mechanical Relations of the Energy of Magnetism," *Roy Soc Proc*, Ser AV, 93, 1916–17; also "The Theory of Electricity," Art 470, p. 417

the same as though no temporary magnetism were present, but in fact it is different, since the magnetic intensity $\boldsymbol{H}$ is different In this case the sources determining the lamellar vector $\boldsymbol{H}$ are not only on the interior and on the boundaries of the permanent magnets, but also on the interior and the boundaries of the temporary magnetic media. If we write I_0 and I for the intensity of magnetization of permanent and temporary magnetism, respectively, equations (5 3) and (5 4) will take the more explicit form

$$-\operatorname{div} \boldsymbol{H} = 4\pi \operatorname{div} (I_0 + I), \quad -\operatorname{divs} \boldsymbol{H} = 4\pi \operatorname{divs} (I_0 + I)$$

which indicates clearly just where these sources are The magnetic induction $\boldsymbol{B}$, given by equation (6 1), may also be written more explicitly in the form

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi(I_0 + I)$$

or preferably

$$\boldsymbol{H} = \boldsymbol{B} - 4\pi(I_0 + I) \tag{9 2}$$

In this notation the potential energy of the magnetic field under consideration is given by equation (9.1) if we replace I by I_0 in that equation, or

$$W_m = -\tfrac{1}{2}\int I_0\ \boldsymbol{H} d\tau \tag{9.3}$$

This integral may be taken throughout all space, since it vanishes outside the permanent magnetic material.

To put into evidence the local part of the energy we shall multiply equation (9.2) scalarly by the vector $\boldsymbol{B} + 4\pi(I_0 + I)$, this gives

$$\boldsymbol{B}\ \boldsymbol{H} + 4\pi(I_0\ \boldsymbol{H} + I\ \boldsymbol{H}) = B^2 - 16\pi^2(I_0 + I)^2$$

If we multiply this equation by $d\tau$ and integrate throughout all space, we shall obtain the equation

$$4\pi\int I_0\ \boldsymbol{H} d\tau = \int[B^2 - 16\pi^2(I_0 + I)^2]d\tau - 4\pi\int I\ \boldsymbol{H} d\tau$$

since the integral $\int \boldsymbol{B}\ \boldsymbol{H} d\tau$ vanishes for a field occupied

by magnetic material only But the left member of this last equation multiplied by $-\frac{1}{8\pi}$ is the potential energy of the magnetic field, or we have

$$W_m = -\frac{1}{8\pi}\int[B^2 - 16\pi^2(I_0 + I)^2]d\tau + \tfrac{1}{2}\int I\,H d\tau \qquad (9.4)$$

From this result we can read off the energy density at every point of space Throughout free space I_0 and I are zero, so that the field or ethereal energy density is

$$-\frac{B^2}{8\pi} = -\frac{H^2}{8\pi}$$

a result just opposite in sign * to the average energy density of Maxwell and the field energy density given by a number of writers. The negative sign is not only a consequence of a logical deduction, but it is also a necessary feature in a consistent theory of magnetic energy. This fact will appear more clearly when we study the magnetic energy of the electric current. Here also we shall see that the ethereal energy density plays the important rôle, this fact and the way the induction entered the theory is the reason why B is regarded as the fundamental vector in the theory of magnetism.

The terms involving I_0 and I constitute the purely local part of the energy, while the second integral in the right member of (9.4) is the strain energy due to the polarization of the temporary magnetic media Under the influence of the magnetic field, the elementary magnets of this media are displaced from their chaotic, magnetically neutral state. The resistance to this strain becomes active in pulling the temporary magnets back into their original state when removed from the magnetic field It is natural to interpret the terms in I symmetric to I_0 in the potential energy function as purely magnetic energy, and

* It is this discrepancy in sign that is the subject of Livens' criticisms. See Livens, *loc cit*

take the non-symmetric term as the strain energy of temporary magnetism

In the case of a weak magnetic field the simplest assumption that can be made concerning induced magnetism is that I follows the linear law

$$I = \mu' H$$

then

$$H = \frac{1}{\mu}(B - 4\pi I_0)$$

where μ is written for $1 + 4\pi\mu'$ If we form the scalar product of the vector, $B + 4\pi I_0$, with each member of the last equation and integrate throughout all space, as in the previous case, we shall find that

$$W_m = -\frac{1}{8\pi\mu}\int (B^2 - 16\pi^2 I_0^2)d\tau.$$

The total energy density in the temporary magnets is $-\frac{B^2}{8\pi\mu}$. If there is no temporary magnetism in the field this becomes $-\frac{B^2}{8\pi}$, and the last equation reduces to

$$W_m = -\frac{1}{8\pi}\int (B^2 - 16\pi^2 I_0^2)d\tau$$

since μ is unity in free space and also in the permanent magnets

In the temporary magnetic media where I_0 is zero,

$$B = (1 + 4\pi\mu')H = \mu H$$

a relation frequently written If this relation is to hold in general, then μ is a very complicated function of the intensity H

The factor, μ, introduced above and called the *permeability*, permits us to classify various substances with respect to the effects produced on them by a magnetic field The permeability, assumed to be unity in free

space, is unlike the dielectric coefficient in that it may assume negative values. Substances for which this is true are called *diamagnetic*. A medium for which μ is constant and greater than unity is called *paramagnetic*, while a *ferromagnetic* substance is characterized by having its permeability greater than unity and variable. This is a characteristic of the various kinds of iron.

It is well to emphasize the fact that, in the theory of magnetic energy, we are still dealing with average values; even the elementary magnet is composed of many molecules. The question of the nature of magnetism does not arise here. To establish a theory of the *nature* of magnetism on the electron hypothesis, it is necessary to pick the atom to pieces. Then Rowland's experiment demonstrating the existence of a magnetic field accompanying a pure convection current leads us to the conclusion that the relative motion of every bound electron in its orbit constitutes an electric current with an accompanying magnetic field. A confirmation of such magnetic effects is obtained through the more refined measuring instrument, the spectrum. It is due to the fact that div $\boldsymbol{B}$ is everywhere zero, and that on the average the vector $\boldsymbol{H}$ has no vortices in the magnetic media, that we can conclude that the integral $\int \boldsymbol{B}\cdot\boldsymbol{H}d\tau$, taken throughout all space, vanishes if the magnetic field is produced by permanent magnets only.

10. Dielectric Polarization.—We have seen (I, 13.6, 13.7) how the electric potential ϕ was determined from the sources of the intensity $\boldsymbol{E}$; these sources were found to be in those regions where div $\boldsymbol{E}$ and divs $\boldsymbol{E}$ were different from zero. This state happened at the true charges, and all points of a dielectric medium where the dielectric coefficient ϵ was a variable point function, and at an interface of two isotropic dielectrics of different dielectric constants.

Under the electron hypothesis it is easy to understand just what happens when a dielectric is brought into an

electrostatic field The field intensity acting on the dielectric has a tendency to drive all protons in the positive sense and all the electrons in the negative sense along the lines of intensity But since the dielectric is a non-conductor, nearly all electrons are strongly bound to the positive nuclei, and there results a strained condition in the dielectric, a displacement of the positive and negative charges along the lines of intensity into a system of bipoles or double sources This strained condition is hypothecated in the relation

$$D = \frac{\epsilon}{4\pi} E.$$

The potential ϕ, whose negative gradient is the field intensity E, may thus be thought of as due, partly, to a surface layer of true electricity on the conductors, and partly to the polarized dielectric We shall call the contribution to the potential due to the true charges spread over the surfaces of the conductors ϕ_0, so that

$$\phi_0 = \int \frac{\omega d\sigma}{r}.$$

Then the potential due to the dielectric polarization will be $\phi - \phi_0$. We shall write P for the *intensity of polarization*, or simply the *polarization*, and define it as *the resultant moment of the moments of all doublets in unit volume*, just as we defined the intensity of magnetization. Then that part of the potential due to the polarized dielectric will be identical in form with the magnetic potential (4 1), or

$$\phi - \phi_0 = \int_\sigma P \operatorname{grad}_M \frac{1}{r} \, d\tau \qquad (10.1)$$

where the integral is to be taken throughout all dielectric media

We now wish to express the polarization P in terms of the known vectors D and E, already introduced. We can

easily do this if we identify the above equation with the electric potential previously obtained (I, 13.7)

$$\phi = \int_\tau \frac{\rho' d\tau}{r} + \int_\sigma \frac{\omega' d\sigma}{r},$$

where

$$\rho' = \frac{1}{4\pi} \operatorname{div} \boldsymbol{E}, \quad \omega' = \frac{1}{4\pi} \operatorname{divs} \boldsymbol{E},$$

here the surface density ω' includes the surface density of the true charges on the surface of the conductors The potential due to the polarized dielectric thus becomes

$$\phi - \phi_0 = \int_\tau \frac{\rho' d\tau}{r} + \int_\sigma \frac{\omega' d\sigma}{r} - \int_\sigma \frac{\omega d\sigma}{r}$$

which we may write in the form

$$\phi - \phi_0 = \frac{1}{4\pi}\int_\tau \frac{\operatorname{div} \boldsymbol{E} d\tau}{r} + \frac{1}{4\pi}\int_\sigma \frac{\operatorname{divs} \boldsymbol{E} d\sigma}{r} - \int_\sigma \frac{\operatorname{divs} \boldsymbol{D} d\sigma}{r}. \tag{10.2}$$

If we apply the transformation theorem

$$\operatorname{div} \phi \boldsymbol{u} = \phi \operatorname{div} \boldsymbol{u} + \boldsymbol{u} \operatorname{grad} \phi$$

to the integral in (10.1) we shall get the integrals

$$\int_\tau \operatorname{div}_M \frac{\boldsymbol{P}}{r} d\tau - \int_\tau \frac{\operatorname{div} \boldsymbol{P} d\tau}{r}$$

for the right member of that equation. The first of these integrals is transformed by Gauss's theorem into the integral $\int \frac{\boldsymbol{P} \cdot \boldsymbol{n} d\sigma}{r}$, where the integral is to be taken over the boundaries of the media so that equation (10.1) becomes

$$\phi - \phi_0 = -\int_\tau \frac{\operatorname{div} \boldsymbol{P}\ d\tau}{r} + \int_\sigma \frac{\boldsymbol{P}\ \boldsymbol{n} d\sigma}{r} \tag{10.3}$$

When we compare equations (10.2) and (10.3) we see that at all points of a dielectric medium

$$-\operatorname{div} \boldsymbol{P} = -\frac{1}{4\pi} \operatorname{div} \boldsymbol{E}.$$

This differential equation is satisfied by the equation

$$P = -\frac{1}{4\pi} E + D$$

since div D is zero at points of the dielectric if no true charges are present. We have also chosen the D positive so that the polarization vanishes in free space, as it should, since here $D = \frac{1}{4\pi} E$.

At an interface of two dielectrics with different dielectric coefficients, $n\ P$ becomes

$$n_1 \cdot P_1 + n_2 \cdot P_2 = - \text{divs } P$$

If we compare the surface integrals over the surfaces of the dielectrics given in equations (10 2) and (10 3) we see that

$$-\text{divs } P = \frac{1}{4\pi} \text{divs } E.$$

And since we are assuming no real charges in the dielectrics or on their boundaries in general, the surface divergence of D vanishes at an interface between two dielectrics, so that again

$$P = -\frac{1}{4\pi} E + D$$

which satisfies the condition that P vanish in free space.

At the surface of a conductor bordering on a dielectric

$$n\ P = \omega' - \omega$$

or

$$-\text{divs } P = \frac{1}{4\pi} \text{divs } E - \text{divs } D$$

the surface divergence degenerates at the surface of a conductor, since P, E, and D are zero in the conducting medium The last equation may thus be written in the form

$$n\ P = -\frac{1}{4\pi} n \cdot E + n\ D$$

where the unit normal n, at the surface, is directed into the conducting medium

From these results we conclude that the polarization P is always given by the equation

$$P = D - \frac{1}{4\pi} E. \tag{10.4}$$

From equation (10.3) we see that the sources of the vector, E, may be expressed by the equations

$$\rho' = -\operatorname{div} P, \quad \omega' = -\operatorname{divs} P$$

as well as by the forms previously given (I, 13 4).

The fictitious electric density or the polarization effects are thus seen to be due to the constitutive electric charges of the dielectric. In this sense the fictitious density is a true electric density. In the normal state every element of a dielectric is electrically neutral, but when placed in an electric field the electric corpuscles constituting the medium are shifted in such a way that every volume element is charged or becomes a real source for the electric intensity.

EXERCISES

(1) Give a second independent deduction for the potential of an elementary magnet

(2) Use the full form for the potential of a small magnet in deducing the potential for the continuous distribution of magnetism, thus showing how the second and higher order terms drop out in applying the fundamental law of the integral calculus.

(3) Give a direct proof for the convergence of the double layer potential if Φ is constant over the shell

(4) Produce the details of the proof that

$$W_m = \tfrac{1}{2}\int I \operatorname{grad} \psi d\tau,$$

by bringing the doublets up from infinity, as suggested in the text

(5) Furnish the details of the proof that the integral $\int B \cdot H d\tau = 0$ when extended throughout all space, as suggested in Art. 8.

CHAPTER III

CURRENT ELECTRICITY

1 **The Electric Current.**—If the charged plates of a condenser be connected by a copper wire, a current called the *conduction current* will flow through the wire, and in a fraction of a second the condenser will be discharged The sense of the current will be defined as positive from the positive to the negative plate through the conductor, the sense of the current thus agrees with the sense of the lines of intensity, $\boldsymbol{E}$, or the positive sense of the current is the sense in which a positive charge would be driven through the wire

Before the contact is made there is a definite difference of potential between the two plates, or between every two points of the field, defined by the line integral $-\int_{P_1}^{P_2} \boldsymbol{E}\cdot d\boldsymbol{r}$. This difference of potential depends on the initial and terminal points only, it is independent of the path and of the time During the discharge of the condenser, however, there is no difference of potential in any proper sense; the intensity, $\boldsymbol{E}$, becomes a vector point function varying with the time, and the line integral is thus no longer independent of the time In this case, the line integral is called the *electromotive force* If the plates of the condenser are connected with a voltaic cell, the charge on the plates may be maintained constant. In this case a constant or a *steady* current will flow through the wire, the intensity, $\boldsymbol{E}$, will be independent of the time, and the electromotive force again becomes the difference of potential.

Every change in a current affects the field intensity at every point of space The effects of the change are not

immediately discerned at a distant point, for it takes time for a disturbance to travel from the conductor to the point. In many cases this delayed effect may be neglected; this is done by assuming that the adjustment is made so quickly that the current may be regarded as steady at that instant. When such conditions persist the current is termed *quasi-stationary*, and the e.m.f. at a given instant is the difference of potential of the field at that instant.

2. Current Strength and Current Density.—In the process of electrolysis, which is known to every student of elementary physics and chemistry, the electrolyte is ionized. It is continually being decomposed into positively and negatively charged particles or ions. The *anion* or negative ion appears at the *anode*, at the *electrode* or plate where the current enters the electrolyte, while the *cation*, the positive ion, appears at the *cathode* or at the electrode where the current leaves the electrolyte. There is a continual drift of the ions through the electrolyte along the lines of intensity. The positive sense of the current through the electrolyte will evidently be from the anode to the cathode.

To arrive at a more exact definition for current or *current strength*, we shall consider an element, $d\sigma$, of a geometric surface across the electrolyte. If N_1 positive natural unit charges, e, cross the element of surface during the time dt, with a velocity v_1, then this quantity, N_1e, of electricity will be contained in a cylinder whose volume is $dtv_1\, nd\sigma$, v_1dt being a directed element of the cylinder and $d\sigma \cos (v_1, n)$ a right section. If the quantity ρ_1 is the electric volume density of the electricity which crossed the surface element with the velocity v_1, then the quantity of electricity, N_1e, may be given as the product of the volume density and the volume of the cylinder, or

$$N_1e = dt\rho_1 v_1\, nd\sigma.$$

The density, ρ_1, is the number of such charges per unit volume, or if the geometric surface intersects some of the

unit charges, then, more generally, ρ_i is charge per volume. On the other hand, n_i electrons crossing the s surface element with the velocity $-v_i$, during the s time interval, dt, would be equivalent to the same num n_i, of positive unit charges crossing in the positive se Or revising the volume density, ρ_i, to include the additi positive charge that we may include in the cylinder, may say that the total charge crossing the surface elem in positive sense is given by the equation

$$(N_i + n_i)e = dt\rho_i v_i \cdot n d\sigma.$$

If we take into account all quantities of electricity cross the whole surface σ with their various velocities du the time interval dt, and call this quantity dq, then by fundamental law of the integral calculus we shall have

$$dq = dt \operatorname*{Limit}_{n \to \infty} \sum_{i=1}^{n} \rho_i v_i \cdot n d\sigma = dt \int_\sigma \rho v \cdot n d\sigma$$

for the quantity of electricity crossing the surface. If write J for the differential quotient, $\frac{dq}{dt}$, then

$$J = \frac{dq}{dt} = + \int_\sigma \rho v \cdot n d\sigma.$$

The quantity J, or the charge crossing the surface per time, is called the *current strength*, or simply the *cur* across the surface The vector quantity, ρv, from its nature is called the *current density* or (since the charges convected or transported by matter) the convection rent density.

3. The Conduction Current.—On the electron hyp esis, the current through a metallic conductor is a con tion current. It has been pointed out that the proto never dissociated from matter, and the electron, on constitutive theory, like the proton, is a part of all mat So if we had taken the surface σ, across the conductor j

ing the plates of the condenser, we would have arrived at the same result

$$J = \int_\sigma \rho v \cdot n d\sigma \tag{3.1}$$

The *conduction current density*, which we shall indicate by the vector i, is thus seen to be the convection current density through a metallic conductor. It is now quite generally believed that the drift or diffusion of electrons only, through the conductor, constitutes the conduction current. The electrons in the conductor that are free or loosely bound to positive nuclei are driven through the material of the conductor towards the positive plate, these are replaced by the electrons from the negative plate till equilibrium is established. That this is so is evidenced by the fact that the electron is more mobile than the proton and the electron is the same wherever found; so that a drift of electrons from one conductor to another of different material would not change the character of the material.

4. Maxwell's Displacement Current.—If the conductor is a linear conductor, one of uniform cross-section, and if its material is isotropic, the lines of intensity E, or the lines of flow, will be parallel to the axis of the conductor, and the current will spread out uniformly over the cross-sections. This is a natural assumption, since the most direct path from condenser plate to condenser plate through the wire is perpendicular to the cross-sections of the wire. During the discharge of the condenser the current strength, J, across a right section of the conductor, will be just equal to the time rate of discharge of the positive plate, since the flux of the current density, i, will be a maximum for these normal sections.

Thus, if we enclose the positive plate of a parallel plate condenser by a surface σ, which encloses the positive plate only and which cuts the conductor in a right section, then the charge on the positive plate will be given by the equation

$$q = \int_\sigma D \cdot n d\sigma$$

according to our revised electric flux theorem, and

$$\frac{dq}{dt} = \int_\sigma \frac{\partial D}{\partial t}\, n d\sigma$$

is the time rate of discharge of the positive plate. But $\dfrac{dq}{dt}$ is here negative, since $\int_\sigma D\, n d\sigma$ is a decreasing function of the time during the discharge of the positive plate. We thus see that

$$J = -\int_\sigma \frac{\partial D}{\partial t}\, n d\sigma$$

and since i is zero at all points of σ except where this surface intersects the conductor, we may write

$$\int_\sigma i\, n d\sigma = -\int_\sigma \frac{\partial D}{\partial t}\, n d\sigma$$

or

$$\int_\sigma \left(i + \frac{\partial D}{\partial t}\right) n d\sigma = 0 \tag{4.1}$$

Since i is current density, it is natural to interpret $\dfrac{\partial D}{\partial t}$ as current density, it is in fact Maxwell's *displacement current* density. It represents the time rate of change of flux of the displacement through an arbitrary surface element at a point of the dielectric per unit area, or it is the time rate of change of the displacement D at a point of the dielectric.

5. The Total Current a Source Free Current.—If we write C for the current density $i + \dfrac{\partial D}{\partial t}$, and apply Gauss's theorem to equation (4.1), we shall get

$$\int_\sigma C\, n d\sigma = \int_\tau \operatorname{div} C d\tau = 0$$

To satisfy this equation we may assume that div $C = 0$, but this is a sufficient and not a necessary condition. This result, however, leads us to Maxwell's fundamental assumption that *the complete current is everywhere source*

free, or the vector lines of the vector **C** *are closed curves, and* div **C** *is everywhere zero* This is a far-reaching, simplifying assumption, to which Maxwell owed his success, while contemporary continental writers became hopelessly involved

To Maxwell, the conductor and the dielectric constituted a *complete* current circuit, and the conduction current density found its counterpart in the dielectric current density, the sum of the two being the *complete* current density **C**, where

$$\boldsymbol{C} = \boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{5 1}$$

Should a medium support a conduction current, a displacement current, and a convection current at the same time, the complete current density, **C**, will be given by the sum of their respective current densities, or

$$\boldsymbol{C} = \boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t} + \rho \boldsymbol{v}. \tag{5.2}$$

If we regard **i** as a convection current density, then the vector **i** may be omitted or regarded as included in the term $\rho \boldsymbol{v}$. In any case we make the assumption that

$$\operatorname{div} \boldsymbol{C} \equiv 0 \tag{5.3}$$

where **C** is the total current density This general assumption of a source free current will lead to satisfying results in the sequel

6. **Ohm's Law.**—If we regard the drift of the electrons through a conductor as a diffusion process like the diffusion of one gas through another, then the force producing the motion, instead of being proportional to the acceleration, is proportional to the velocity. Under this assumption, we have for an isotropic conductor, since $\boldsymbol{i} = \rho \boldsymbol{v}$,

$$\boldsymbol{i} = \kappa \boldsymbol{E} \tag{6.1}$$

where the constant κ, called the conductivity, is a constant of the material of the conductor and subject to experi-

mental determination. Equation (6.1) is usually referred to as the differential form of Ohm's law.

To obtain the integral form we will multiply equation (6.1) scalarly by the vector $d\sigma n$ or $nd\sigma ds$, where n is the unit normal to the surface element $d\sigma$. If we integrate throughout a segment of the conductor between the normal sections through the points P_1 and P_2, P_1 being a point of lower potential than P_2, we find that

$$\int\int i \cdot nd\sigma ds = \int\int \kappa E \cdot nd\sigma ds.$$

If we assume that the current is steady and the conductor isotropic, then for a conductor of uniform cross section the order of integration is immaterial. We may thus write the above double integrals as iterated integrals and in the forms

$$\int_{P_1}^{P_2} ds \int_{\sigma} i \cdot nd\sigma = \kappa \int_{\sigma} d\sigma \int_{P_1}^{P_2} E \cdot nds$$
$$= \kappa \int_{\sigma} d\sigma \int_{P_1}^{P_2} E \cdot dr$$

where we have written dr for its equal dsn. The first surface integral is the current strength J, a constant, so that the left member becomes $-sJ$. The negative sign is due to the fact that the line integral from P_1 to P_2 is in a sense just opposite to the sense of the current, so if s is taken positive from P_2 to P_1, which is the positive sense of the current, then it is negative when measured from P_1 to P_2. Since $E = -\text{grad}\ \phi$, it will be clear that the right member is $-\kappa\sigma(\phi_2 - \phi_1)$, or upon equating these results we find that

$$\phi_2 - \phi_1 = \frac{s}{\kappa\sigma} J$$

or

$$\phi_2 - \phi_1 = RJ \tag{6.2}$$

when we write R for $\frac{s}{\kappa\sigma}$.

Equation (6.2) is the integral form sought The new factor, R, is called the resistance It is seen to be proportional to the length of the segment of the conductor and inversely proportional to the product of the conductivity and the cross-sectional area. It also depends on the character of the material, and like the conductivity, it may be experimentally determined The factor $\frac{1}{\kappa}$ is sometimes referred to as the *resistivity*.

The integral form of Ohm's law is usually given as an empirical formula, its validity has been very carefully tested. The differential form is applicable to every type of conduction current and also to eolotropic conductors. If the material of the conductor is eolotropic the conductivity may be a homography, and the current density a linear vector function of the intensity.

7. Kirchhoff's First Law.—If we apply the assumption of a source free current to the situation considered in equation (5.1), then

$$\operatorname{div} C \equiv \left(\operatorname{div} i + \frac{\partial \rho}{\partial t}\right) = 0 \tag{7.1}$$

where we have replaced div D by ρ, the electric volume density. For a steady current flowing in an isotropic linear conductor the displacement D, and therefore the electric volume density ρ, will be independent of the time, so that equation (7.1) reduces in this case to

$$\operatorname{div} i = 0.$$

This is Kirchhoff's first law. It expresses the facts that *in case of a steady current there is no piling up of either positive or negative electricity in any volume element, and that the current strength, J, across every section of the conductor is the same.* These are the results to be expected from a steady current, in fact, if there were a piling up of electric charges in some volume element of the conductor, the intensity, E, would change, and the current would cease

to be steady We may also conclude that the lines and tubes of intensity outside the conductor, which terminate where the density $\rho \neq 0$, must terminate at the surface of the conductor Experiment shows that this external field is static and takes no part in conduction.

We have seen (I, 14.1) that at an interface

$$|E_1| \sin \alpha_1 = |E_2| \sin \alpha_2$$

So if E_2 is the intensity just inside the surface of the conductor, E_2 will be parallel to the surface, and the angle α_2 will be $\frac{\pi}{2}$, and therefore $|E_1| \sin \alpha_1 = |E_2|$. So in general the intensity, E_1, is oblique to the surface of the conductor. For a perfect conductor, E_1 could be normal to the surface, in this case the resistance would be zero and the conductivity infinite Also since

$$E_2 = -\frac{i}{\kappa} = 0$$

the angle α_1 must vanish, making E_1 normal to the surface of the conductor in this case Since the displacement and the electric intensity are parallel for an isotropic dielectric, the displacement lines and tubes will in general be oblique to the surface of the conductor.

For a variable current, equation (7.1) operates, or in every volume element there is an electric volume density changing with the time, as we might expect.

8 The First Fundamental Law of Electrodynamics.— In 1820 Oersted discovered that the electric field of the electric current, unlike the electrostatic field, is always accompanied by a magnetic field The lines of magnetic intensity, H, are closed curves threading the circuit. The sense may be determined by grasping the conductor with the right hand, the thumb in the positive sense of the current; the fingers will then be pointing in the positive sense of the lines of magnetic intensity The experiments and discoveries of Oersted were interpreted three years

later by Ampère in his fundamental *mémoire* on electro-dynamics.*

His principal deduction may be stated as follows:

If a current flows in a small plane circuit, the magnetic field produced at points whose distances from the circuit are large compared to the dimensions of the circuit may be reproduced by a small magnet with its center in the plane of the circuit and its axis perpendicular to the plane, and whose strength is proportional to the product of the strength of the current and the area of the circuit. The substitution becomes the closer the smaller the circuit.

If we choose to measure the current by its magnetic effect, we may define unit current as that current flowing in a circuit of unit area which produces at its center a magnetic moment of one unit. Thus one (c. e. m.) unit current is a current flowing in a circuit of one square centimeter area producing one (c. e. m.) unit moment.

The above deduction by Ampère may be applied to every current circuit by converting the circuit into a system of mesh circuits, i.e., into a network of conductors having the original circuit for the boundary of the net. The currents in the mesh circuits are assigned the same strength, J, as that of the original current. That the system of mesh currents is equivalent to the original current is seen from the fact that all mesh currents will cancel off, since every segment of a mesh plays the rôle of conductor for two currents of equal strengths but of opposite sense, thus leaving the original current, J, in the original circuit. According to Ampère's principle, each mesh current may be replaced by an elementary magnet of strength J whose center is in some surface containing the net and whose axis is normal to this surface. As the number of meshes is increased without limit, the area of the surface bounded by each mesh circuit approaches a plane area as its limit, and the distribution of magnets replacing each successive

* "Théorie des phénomènes électro dynamiques," *Mémoires de l'Institut*, IV, 1823.

mesh system approaches a magnetic shell of strength J as its limit. We are thus led to Ampère's fundamental theorem

The magnetic field produced by an electric current flowing through a linear conductor may be produced by a magnetic shell whose boundary coincides with the conductor and whose strength is equal to the strength of the current measured in electromagnetic units.

Such a shell and its corresponding circuit will be spoken of as equivalents

We found that the magnetic potential

$$\psi = \int_{\infty}^{P} \operatorname{grad} \psi \, dr = -\int_{\infty}^{P} H \, dr$$

is a unique single valued function of the point P, independent of the path of integration So if the integral is taken from the point P, around a contour back to the point P again, the contour integral is zero, even if the path of integration passes through the shell But we have seen that if the strength of the shell is J, this contour integral increases by $4\pi J$ in passing through the shell in the positive sense, that is, in the positive direction of the lines of magnetic intensity Thus if the shell is replaced by its equivalent circuit the integral $\int_s \operatorname{grad} \psi \, dr$, taken around a contour in positive sense and threading the circuit once, is $4\pi J$ less than the previous contour integral whose path intersected the equivalent shell; or for a path threading the circuit once

$$\int_0 \operatorname{grad} \psi \, dr = -\int_0 H \, dr = -4\pi J$$

From this we have the relation

$$\int_0 H \, dr = 4\pi J \tag{8.1}$$

which expresses the fact that the work done by the magnetic field in driving a positive unit magnetic pole around

a current once is just equal to 4π times the strength of the current measured in (c e m.) units Equation (8.1) is frequently called the *first law of circulation.*

In the case of a magnetic shell the field intensity, H, was the negative gradient of a single valued potential function, in which case curl $H \equiv 0$. This result depends on the fact that the circulation integral $\int_0 H\ dr$ is independent of the path of integration. If the magnetic shell be replaced by its equivalent circuit, curl H will no longer be identically zero, H will still be the negative gradient of a scalar point function, but not a single valued point function. The circulation integral $\int_0 H\ dr$, is no longer independent of the path of integration, it increases or decreases by $4\pi J$ every time the path of integration threads the current circuit Thus the multiple valued potential due to the current circuit is the potential due to the equivalent shell plus $4\pi nJ$, where n is some positive or negative integer.

If we apply Stokes' theorem to the left member of the last equation and write the integral $\int_\sigma i\ n d\sigma$ for the current strength J, equation (8.1) will take the form

$$\int \operatorname{curl} H \cdot n d\sigma = 4\pi \int_\sigma i \cdot n d\sigma.$$

If the contour bounding the surface σ be contracted to coincide with the boundary of the element $d\sigma$, at a point in the conductor, the above relation will still persist for the element, and since the orientation of the element is as arbitrary as the surface

$$\operatorname{rot} H = 4\pi i$$

which is Ampère's law. If the current J had been measured in (c. e. s.) units, then the circulation integral would have been proportional to the current strength J multi-

plied by 4π. If we introduce the proportionality factor $\frac{1}{c}$ (8 1) will become

$$\int_{\sigma} \operatorname{rot} H \, n d\sigma = \frac{4\pi}{c} J \tag{8.2}$$

and Ampère's law will be given by the equation

$$\operatorname{rot} H = \frac{4\pi}{c} i \tag{8.3}$$

where the factor $\frac{1}{c}$ then depends on the choice of units

This result is perfectly consistent, for we are considering a steady current in a linear conductor. In this case we have seen (Art 6) that div i is identically zero, this, then, accords with the fact that the left member of equation (8 3) is curl H, a solenoidal vector whose divergence is always zero If, on the other hand, the conduction current is variable, the conduction current is not the total current, and the vector i is no longer solenoidal. Equation (8 3) in this case would not be tenable But we have seen that the conduction current was completed by the displacement current, and that the conduction current and the displacement current together constituted the complete source free current. We thus generalize Ampère's law and write

$$\operatorname{curl} H = \frac{4\pi}{c}\left(i + \frac{\partial D}{\partial t}\right) \tag{8.4}$$

This equation is consistent since each member is a solenoidal vector, the left member identically so and the right member on the assumption of the complete or total current being source free Equation (8 4) is the first fundamental equation of the Maxwell theory for stationary media It is the natural generalization of Ampère's law and the one Maxwell made.

In applying this generalization to an arbitrary surface element, $d\sigma$, we will have the work done in carrying a positive magnetic pole once around the boundary of the

element proportional to the total current flowing through the element. If there is a conduction current only, through the element, equation (8.4) reduces to Ampère's law. If the element is outside a conductor the generalization reduces to Ampère's law applied to a displacement current through the element. If the conduction current is steady the displacement current through the surface element outside the conductor vanishes, and so also does the work of carrying a magnetic pole around the element, which agrees with the fact that the intensity, E, is a potential vector in a static electric field.

9. Power and the Impressed Electromotive Force.—To perpetuate the motion of a mechanical system, unless the system is ideal, outside forces called impressed forces must be applied to the system to replace the energy losses due to friction. The situation is very similar in the case of the electric current, for the electric field of the electric current is not only accompanied by a magnetic field, but there is also a dissipation of energy in the conductor in the form of heat. If we think of the mechanical interpretation of heat we see in the diffusion of the electrons and their collisions with the molecules of the conductor an explanation for this transformation of energy of the electric current into unavailable heat energy.

In order to conserve a current in a conductor by replacing the energy dissipated in heat and by supplying the energy used up in work done by the electric current, energy must enter the circuit from an outside source. The *power* per unit current or *the time rate per unit current with which energy enters the current circuit is called an impressed electromotive force.* The maintenance of the current is due to this electromotive force. If the agent (the dynamo-voltaic cell) transforming energy of some other kind into the energy of the electric current is capable of the reverse process of transforming electrical into mechanical energy, the e.m.f. is called an intrinsic e.m.f. This impressed electromotive force produces an impressed electric intensity,

E, in the conductor The vector lines of the vector, **E**, are stream lines or lines of flow.

The work done in driving a positive electric charge through a conductor from a point P_2 to a point P_1 of lower potential is

$$W = q\int_{P_2}^{P_1} \mathbf{E}\, dr = -\, q\int_{P_1}^{P_2} \mathbf{E}\, dr.$$

But this is the necessary energy supplied to do this work. Thus the power per unit current or time rate per unit current with which energy is supplied is

$$\frac{1}{J}\frac{dW}{dt} = \frac{dW}{dt}\cdot\frac{1}{\frac{dq}{dt}} = \frac{dW}{dq} = -\int_{P_1}^{P_2} \mathbf{E}\cdot d\mathbf{r}$$

which identifies the impressed electromotive force with the electromotive force as previously defined

10. Joule's Law.—If a steady current is flowing in an isotropic conductor and just enough energy is supplied to cover the dissipation in heat, then since the energy supplied will be just equal and opposite in sense to the energy, H, dissipated in heat, we may write

$$\frac{1}{J}\frac{dW}{dt} = -\frac{1}{J}\frac{dH}{dt} = -\int_{P_1}^{P_2} \mathbf{E}\cdot d\mathbf{r}$$

If we apply Ohm's law to the right member of this equation we find that

$$\frac{1}{J}\frac{dH}{dt} = -\,(\phi_2 - \phi_1) = -\,JR$$

or

$$\frac{dH}{dt} = -\,J^2R \qquad (10.1)$$

which is the usual form for Joule's law. This law was first deduced experimentally by Joule and formulated correctly by him. We may state the result as follows *If a steady current is flowing in an isotropic conductor the time rate of dissipation of energy in the form of heat in the conductor is*

equal to the product of the resistance and the square of the current strength.

The counter electromotive force, $-JR$, is a passive e.m.f. in that it has no independent existence of its own, it comes into being by virtue of an impressed e.m.f., and in mechanics corresponds to a passive resistance. It is also non-intrinsic since the dissipated energy is not capable of being transformed back into the energy of the electric current.

To deduce the differential form of Joule's law we apply equation (10.1) to a volume element, an elementary current tube of length ds and cross-section $d\sigma$. If we use Newton's fluxion notation we shall have

$$d\dot{H} = -(\mathbf{i}\cdot\mathbf{n}d\sigma)^2\frac{ds}{\kappa d\sigma} = -\frac{i^2d\sigma^2ds}{\kappa d\sigma}$$
$$= \frac{1}{\kappa}i^2d\sigma ds$$

or if we replace $\mathbf{i}$ by its equal $\kappa\mathbf{E}$ we may write

$$d\dot{H} = -\kappa E^2d\tau \tag{10.2}$$

or the time rate of dissipation per unit volume is $-\kappa E^2$. This form of Joule's law is applicable to eolotropic conductors and to every kind of conduction current.

11. Kirchhoff's Second Law.—For completeness we should consider another important law due to Kirchhoff. It states the fact that for a network of conductors, connected in series and carrying steady currents, the sum of the intrinsic e.m.f.'s in the segmental circuits is equal to the sum of the products of each current by the resistance in the circuit segment carrying the current.

We shall suppose that the circuits are connected at points, 1, 2, 3 . . . n; we may then designate the intrinsic e.m.f., the voltage, the current, and the resistance in the circuit $i+1, i$, by $\phi'_{i+1,i}$, $\phi_{i+1,i}$, $J_{i+1,i}$ and $R_{i+1,i}$ respectively. Thus for this circuit we shall have

$$\phi'_{i+1,i} + \phi_{i+1,i} = R_{i+1,i}\, J_{i+1,i}$$

If we sum this equation for all circuits of the net we shall get

$$\Sigma\phi' = \Sigma RJ$$

since

$$\Sigma\phi = \int_s E\ dr$$

and the line integral, being our potential function which is independent of this path of integration, is identically zero for a closed contour This then completes the proof of Kirchhoff's second law

12. The Induced Electromotive Force.—Faraday found that if the lines of magnetic induction are cut by a conductor an electric current will flow through the conductor. Such a current is called an *induced current* The electric intensity set up in the conductor is called an *induced electric intensity*, and the corresponding electromotive force an *induced* e.m f Since in this case energy is transferred rather than transformed the induced e m f. is non-intrinsic.

13. Faraday's Law—Faraday's experiments led him to the conclusion that the induced electromotive force necessary to drive the current in the circuit depended on the change in the flux of the lines of induction through the circuit The results obtained by Faraday and summarized by him may be stated as follows

The induced electromotive force set up in a conductor is equal to the time rate of decrease in the number of lines of induction threading the circuit

The number of lines of induction threading the circuit is just equal to the number passing through an arbitrary surface having the conductor for its boundary. But this number, N, is defined by the equation

$$N = \int_\sigma B\ nd\sigma$$

where the integral is the total flux of induction through the arbitrary surface having the conductor for its boundary. The number N, so defined, is independent of the choice of

the surface σ; for the flux is the same through every such surface since div B is identically zero

Conforming to the ideas and experiments of Faraday we define the induced electromotive force as *the time rate of decrease in the number of lines of induction threading the current circuit*, or

$$-\frac{\partial N}{\partial t} \equiv -\int_\sigma \frac{\partial B}{\partial t}\, n d\sigma.$$

This then is the time rate per unit current with which energy enters the circuit So if we write the vector E for the induced electric intensity induced in the circuit, and measure this intensity in electromagnetic units, the induced electromotive force already defined will also be given by the contour integral $\int_0 E\, dr$, taken around the circuit so that

$$\int_0 E\, dr = -\int_\sigma \frac{\partial B}{\partial t}\, n d\sigma. \tag{12.1}$$

If the intensity, E, is measured in electrostatic units, and the induction, B, in electromagnetic units, we must introduce a proportionality factor, $\frac{1}{c'}$, and write

$$\int_s E\, dr = -\frac{1}{c'}\int_\sigma \frac{\partial B}{\partial t}\, n d\sigma. \tag{12.2}$$

If we transform the left member of this equation into a surface integral, by Stokes' theorem, we may write it in the form

$$\int_\sigma \text{curl}\, E\, n d\sigma = -\frac{1}{c'}\int_\sigma \frac{\partial B}{\partial t}\, n d\sigma \tag{12.3}$$

Since the surface σ is arbitrary, these integrals are identical, so that

$$\text{curl}\, E = -\frac{1}{c'}\frac{\partial B}{\partial t} \tag{12 4}$$

This is the differential form of Faraday's law and the second fundamental equation of the electromagnetic theory This equation is self-consistent since div B is identically zero.

In the electrostatic field E is $-\text{grad}\ \phi$, and curl E is therefore identically zero But in a variable electric field we see (12 4) that in general curl $E \neq 0$. Thus the circulation integral $\int_0 E\ dr$ is different from zero.

That equation (12 4) holds for every surface element, $d\sigma$, is a natural assumption to make, for if the induction, B, is varying at the element, the e.m f around the element would be different from zero if a conductor were there, since a current would then flow through the conductor. It is thus assumed that Faraday's law holds for every surface element

13 The Determination of the Universal Constants c and c'—If we start with the electromagnetic unit pole as defined in Art 8, we may express all magnetic quantities in common electromagnetic units, just as we were able to express all electrical quantities in electrostatic units. The two systems of units, as we shall see, are related through the definition for the electromagnetic unit current strength.

If we measure all magnetic quantities in free space or in non-magnetic media, the dimensional equation for magnetic pole strength, m, will be given by the dimensional equation

$$[m] = [F^{1/2}L] = [M^{1/2}L^{3/2}T^{-1}]$$

since the magnitude of the repelling force, F, between two like magnetic poles of strength m, is $\frac{m^2}{r^2}$ We are here using $[F]$ for the derived dimensional unit force.

When we wish to emphasize the fact that a quantity, J, is measured in electrostatic or electromagnetic units, we will write it with a subscript· e in the former case, and m in the latter Thus Ampère's law (8.1), when the cur-

rent, J, is expressed in electrostatic units, may be written in the form

$$\int_0 H\,dr = \frac{4\pi}{c} J_e \tag{13.1}$$

where $\frac{1}{c}$ is the constant of proportionality already introduced in the differential form of Ampère's law. The left member of equation (13.1) is work per unit pole strength, expressed dimensionally it is

$$\frac{[FL]}{[F^{1/2}L]} = [F^{1/2}]$$

The factor $J_e = \frac{dq_e}{dt}$ in the right member is given dimensionally by the equation

$$[J_e] = [F^{1/2}LT^{-1}]$$

so that

$$[c] = [LT^{-1}]$$

or the constant c has the dimensions of velocity.

If, now, we divide equation (8.1) by (8.2) we will find that

$$c = \frac{J_e}{J_m} = \frac{dq_e}{dq_m} = \frac{q_e}{q_m} \tag{13.2}$$

Thus if an electric charge be measured in electrostatic units and the same charge in electromagnetic units, the quotient of the former by the latter will furnish the numerical value of the constant c. This universal constant has been very carefully determined, its value is estimated to be the velocity of light in free space, 3×10^{10} cm./sec.

If we use Faraday's law in its integral form given by equation (12.2), and write it out dimensionally, we may show, just as in the case of the constant c, that c' also has the dimensions of velocity. If we multiply equations (12.1) and (12.2) by q_m and q_e, respectively, where q_m and q_e represent the same charge expressed numerically in different units, the left members of the resulting equations will

each be work expressed in ergs They are thus equal, so if we subtract these two resulting equations we will find that

$$c' = \frac{q_e}{q_m}$$

which identifies the constants c and c'

The relation (13 2) is a consequence of our definition for electromagnetic unit current This equation indicates the relation existing between the two sets of units. The velocity of light acts as a modulus for changing quantities measured in electromagnetic units into electrostatic units

14. Maxwell's Equations.—In the previous sections we deduced the electromagnetic field equations of the Maxwell theory for stationary media. We found these equations in the form

$$\left.\begin{aligned} \operatorname{curl} \boldsymbol{H} &= \frac{4\pi}{c}\left(i + \frac{\partial \boldsymbol{D}}{\partial t}\right) \\ \operatorname{curl} \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \\ \operatorname{div} \boldsymbol{D} &= \rho \\ \operatorname{div} \boldsymbol{B} &= 0 \end{aligned}\right\} \qquad (14.1)$$

wherein all electric quantities are to be measured in (c e.s) units and all magnetic quantities in (c e.m) units These are Maxwell's equations They are inadequate for the determination of the electromagnetic field at every instant, t But for an isotropic medium in which a linear relation exists between the intensity of magnetization, I, and the magnetic intensity, H, we have the additional conditions

$$\left.\begin{aligned} \boldsymbol{D} &= \frac{\epsilon}{4\pi}\boldsymbol{E} \\ \boldsymbol{B} &= \mu \boldsymbol{H} \\ i &= \kappa \boldsymbol{E} \end{aligned}\right\} \qquad (14\ 2)$$

where the constants ϵ, μ, and κ are to be determined experimentally.

These equations, (14.1) and (14.2), are suitable for the solution of many problems, but they are very much restricted in their applicability by the assumption that ϵ, μ, and κ are numerical constants. And then in their determination Maxwell recognized the fact that their values, determined by the use of a steady electromagnetic field, would not be applicable to a rapidly changing state, as in the case of light.

Maxwell not only computed the value of the universal constant c, but also predicted that light waves were electromagnetic. This prediction of Maxwell was confirmed by Hertz in a series of laboratory experiments in which he produced electromagnetic waves which were reflected, refracted, and polarized, thus showing that electromagnetic waves were transverse waves, and except for a difference in wavelength were analogous to light waves.

If we substitute equations (14.2) into (14.1) we obtain the field equations

$$\left.\begin{aligned} \operatorname{curl} H &= \frac{1}{c}\left(4\pi\kappa E + \epsilon\frac{\partial E}{\partial t}\right) \\ \operatorname{curl} E &= -\frac{\mu}{c}\frac{\partial H}{\partial t} \\ \operatorname{div} \mu H &= 0,\ \operatorname{div} \epsilon E = 0, \end{aligned}\right\} \qquad (14.3)$$

the form given by Heaviside and Hertz. In a medium where there is no conduction current present the electromagnetic field will be defined by equations (14.3) after the conduction current κE is put equal to zero. The equations then assume the simpler form

$$\left.\begin{aligned} \operatorname{curl} H &= \frac{\epsilon}{c}\frac{\partial E}{\partial t} \\ \operatorname{curl} E &= -\frac{\mu}{c}\frac{\partial H}{\partial t} \\ \operatorname{div} H &= 0, \quad \operatorname{div} E = 0 \end{aligned}\right\} \qquad (14.4)$$

for media where ϵ and μ are constants.

When we take the curl of each member of the first of equations (14.4) and substitute for curl E from the second we find that

$$-\text{rot rot } H = \frac{\epsilon\mu}{c^2}\frac{\partial^2 H}{\partial t^2}$$

And since div H is identically zero, if μ is constant,

$$-\text{rot rot } H \equiv \text{grad div } H - \text{rot rot } H$$

so that we may write the above equation in the form

$$\Delta' H = \frac{\epsilon\mu}{c^2}\frac{\partial^2 H}{\partial t^2}. \tag{14.5}$$

In exactly the same way we find that the electric intensity, E, satisfies the same equation, or

$$\Delta' E = \frac{\epsilon\mu}{c^2}\frac{\partial^2 E}{\partial t^2} \tag{14.6}$$

These differential equations (14.5) and (14.6), define the electromagnetic field at all points of the media; at a bounding surface where ϵ or μ or both change their values, the electric and magnetic intensities satisfy special boundary conditions (I, 14 1, II, 7 4) as we have seen before. These equations are also typical wave equations The x coordinate, E_x, of the intensity E evidently satisfies the equation

$$\Delta E_x = \frac{\epsilon\mu}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

and the other coordinates of the vectors E and H are similarly defined. If we assume that E_x is independent of the coordinates y and z, and write $\frac{1}{a^2}$ for the constant $\frac{\epsilon\mu}{c^2}$, the defining equation for this coordinate will be

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2\theta}{\partial t^2} \tag{14.7}$$

The well-known solution of this equation is

$$\theta = f_1\left(t - \frac{x}{a}\right) + f_2\left(t + \frac{x}{a}\right) \tag{14.8}$$

the functions f_1 and f_2 being arbitrary.

If we consider the special solution,

$$\theta = f\left(t - \frac{x}{a}\right) \tag{14.9}$$

(and that this is a solution of (14.7) is easily verified by a direct substitution), it may be shown that this is a wave propagated along the axis OX in positive sense with the velocity of propagation equal to a. In fact, the necessary and sufficient condition that the function θ, given by equation (14.7), is a wave propagated along the axis OX in positive sense, is that the quantity a is the velocity of propagation along this axis in positive sense

To show the necessity of the condition, we will write the ordinate at the time $t = t_1$, corresponding to the abscissa $x = x_1$; if we use a corresponding subscript in writing the ordinate we will have

$$\theta_1 = f\left(t_1 - \frac{x_1}{a}\right)$$

If during the time δt, the wave has traveled in positive sense through the distance δx, the time t will be $t_1 + \delta t$, and the new x coordinate $x_1 + \delta x$, so that the ordinate θ_2 will now be given by the equation

$$\theta_2 = f\left(t_1 + \delta t - \frac{x_1 + \delta x}{a}\right)$$

But if the wave is propagated without change of form these two ordinates, θ_1 and θ_2, are identical, and this is seen to be so if for every arbitrary time interval

$$\delta t = \frac{\delta x}{a}$$

or since $a\delta t$ is the space, δx, passed over, the quantity a is the velocity of propagation.

To prove the sufficiency of the condition, we need only show that every ordinate, θ, does not change with the time, when the quantity a is the speed with which the corresponding abscissa increases; the speed a is here then the velocity

of propagation At the instant t_1, the ordinate θ_1 will be given in terms of its corresponding abscissa, x_1, by the equation

$$\theta_1 = f\left(t_1 - \frac{x_1}{a}\right)$$

as before If during the time δt, the abcissa x_1 has increased by δx, we will have in general a new θ defined by the equation

$$\theta_2 = f\left(t_1 + \delta t - \frac{x_1 + \delta x}{a}\right)$$

But since the speed a is the velocity of propagation,

$$\delta x = a\delta t.$$

This result substituted in θ_2 identifies it with θ_1, or every ordinate, θ_1, travels along the axis OX, in positive sense, with a velocity a, without change of magnitude. Thus the whole wave is propagated in the positive sense with velocity a, without change of form

We must also regard the equation

$$\theta = f\left(t + \frac{x}{a}\right) \equiv f\left(t - \frac{x}{-a}\right)$$

as a wave propagated along the axis OX, with the velocity of propagation equal to $-a$. Equation (14.8) then represents two waves traveling in opposite senses. Should the wave form repeat itself periodically, or, what is the same thing, if the function f is a periodic function, possibly a sine or cosine function, then at the point x_1 of the OX axis, the ordinate θ_1 will repeat itself periodically. Physically we would say, at the point x_1, the electromagnetic disturbance is a periodic disturbance or a train of waves A discussion of the solution of the more general equations (14.5) and (14 6) will be considered in a later chapter; however, sufficient has been done to show that the theory predicts that electromagnetic disturbances are propagated through material media, with a velocity equal to $\frac{c}{\sqrt{\epsilon\mu}}$. In free

space where $\epsilon = \mu = 1$, this velocity reduces to the velocity of light. The extent to which these results are applicable will also be left to a subsequent chapter.

EXERCISES

(1) Start with the integral form of Ohm's law and show how the differential form may be deduced.

(2) Write out Maxwell's equations for free space.

(3) Show how Maxwell's equations involve electrostatics as a special case.

(4) By a direct substitution show that

$$\phi = f_1\left(t - \frac{x}{a}\right) + f_2\left(t + \frac{x}{a}\right)$$

is a solution of

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 \phi}{\partial t^2}$$

(5) Discuss the fact that the equations

$$E_y = a \cos n\left(t - \frac{x}{c}\right), \quad H_z = a \cos n\left(t - \frac{x}{c}\right)$$

$$E_x = E_z = H_x = H_y = 0$$

represent a polarized plane wave. Why a wave? Why a plane wave? Why polarized?

(6) What is the basic assumption in current electricity?

CHAPTER IV

THE DYNAMICS OF THE ELECTRIC CURRENT

1. D'Alembert's Principle and the Energy Equation.—If we consider a system of particles of mass m_i ($i = 1 \ldots n$), instead of a single particle as in Chap. I, Art. 7, we will be led through D'Alembert's principle to one of the most fundamental theorems of mechanics, namely, Hamilton's principle. If F_i and S_i are the resultants of the applied forces and the constraining forces on the particle m_i, respectively, the equation of motion for the particle may be written in the form

$$m_i \ddot{r}_i - F_i = S_i \tag{1.1}$$

D'Alembert's insight into mechanical systems led him to observe that all *virtual* or *possible displacements* of the system (i.e., those consistent with the constraints) were in general perpendicular to the constraining forces. For example, if a particle is constrained to move on a perfectly smooth fixed surface the constraining force is normal to the surface, and every virtual displacement of the particles is tangential to the surface. Thus if δr_i is a virtual displacement of the particle m_i, this observation of D'Alembert's called *D'Alembert's principle*, or the *principle of virtual work*, is expressed analytically for the particle by the equation

$$S_i \cdot \delta r_i = 0.$$

If we sum for the whole system of particles, D'Alembert's principle takes the analytic form

$$\sum_{i=1}^{n} m_i \ddot{r}_i \cdot \delta r_i - \sum_{i=1}^{n} F_i \cdot \delta r_i = 0. \tag{1.2}$$

Thus the constraints, S_i, are really "lost forces," and disappear from the equation of virtual work.

If we use the actual displacement, dr_i, and integrate equation (1.2) between the limits t_0 and t, we obtain the equation

$$\int_{t_0}^{t}\sum_{i=1}^{n} m_i \ddot{r}_i \cdot \dot{r}_i dt = \int_{t_0}^{t}\sum_{i=1}^{n} F_i \cdot dr_i. \tag{1.3}$$

If we interchange the signs of summation and integration and perform the integration in the left member, we will have the equation of kinetic energy and work in the form

$$\sum_{i=1}^{n} \tfrac{1}{2} m_i \dot{r}_i^2 \Big|_{t_0}^{t} = \sum_{i=1}^{n} \int_{t_0}^{t} F_i \cdot dr_i. \tag{1.4}$$

For a conservative system of forces,

$$F_i = -\operatorname{grad} W_i$$

and if we write T for the kinetic energy of the system, following the notation of I, Art. 7, equation (1.3) becomes the energy equation and may be written in the form

$$T - T_0 = -W + W_0$$

where

$$W = \Sigma W_i$$

is the potential energy of the system, or

$$T + W = T_0 + W_0 = h$$

a constant. We may observe again that the potential energy, W, is the negative of the work done by the conservative forces in an arbitrary displacement of the system.

2. Hamilton's Principle.—To deduce Hamilton's principle we multiply the identity

$$\frac{d}{dt}(\dot{r}_i \cdot \delta r_i) = \ddot{r}_i \cdot \delta r_i + \dot{r}_i \cdot \delta \dot{r}_i$$
$$= \delta \tfrac{1}{2}\dot{r}_i^2 + \ddot{r}_i \cdot \delta r_i$$

by $m_i dt$, sum for all particles, and integrate between the limits t_0 and t_1. We find that

$$\Sigma m_i \dot{r}_i \cdot \delta r_i \Big|_{t_0}^{t_1} = \int_{t_0}^{t_1} (\delta T + \Sigma F_i \cdot \delta r_i) dt$$

where we have replaced $\sum_{i=1}^{n} \frac{1}{2} m_i \dot{r}_i^2$ by T and $\sum_{i=1}^{n} m_i r_i \cdot \delta r_i$ by its equal as given by equation (1.2)

During the time interval $t_1 - t_0$, the particles travel along a set of paths from an initial to a final state; the vector δr_i represents a displacement of the path of the ith particle. Now if the initial and final positions of the system of particles are fixed, the displacements of the end points of the paths are identically zero. The arbitrary displacements δr_i, of the particles m_i, at the instants t_0 and t_1, are the displacements of the end points of the paths, and since these displacements are zero, the left member of the last equation vanishes and we have the equation

$$\int_{t_0}^{t_1} (\delta T + \sum_{i=1}^{n} F_i \cdot \delta r_i) dt = 0 \tag{2.1}$$

This is Hamilton's principle in its most general form. For a conservative system of forces

$$\sum_{i=1}^{n} F_i \cdot \delta r_i = - \delta W$$

so that in this case Hamilton's principle has the form

$$\delta \int_{t_0}^{t_1} (T - W) dt = 0$$

If we replace $T - W$ by L, called the ***kinetic potential*** or the Lagrangian function, the principle assumes the simpler form

$$\delta \int_{t_0}^{t_1} L dt = 0. \tag{2.2}$$

This principle of Hamilton may be stated as follows:

If the initial and final configurations of a mechanical

system are fixed, the time integral of the kinetic potential is stationary.

This broad principle holds for non-conservative systems also, but in this case the variation of the kinetic potential must be replaced by $\delta T + \Sigma F_i \delta r_i$.

3. Lagrange's Equations.—Out of Hamilton's principle (2.2) we can easily deduce Lagrange's equations of motion in generalized coordinates. We define a set of generalized coordinates in the following way: If the position of a point is given by the numbers $(q_1, \ldots q_s)$, then these numbers are called the generalized coordinates of the point. In this case r_i will be a function of the $q_j (j = 1 \ldots s)$.

The kinetic energy of a system of n particles in terms of the q_j will evidently be

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2 = \tfrac{1}{2}\sum_{i=1}^{n} m_i\left(\sum_{j=1}^{s}\frac{\partial r_i}{\partial q_j}\dot{q}_j\right)^2 = \sum_{jk=1}^{s} Q_{jk}\dot{q}_j\dot{q}_k.$$

where

$$Q_{jk} = \tfrac{1}{2}\sum_{i=1}^{n} m_i \frac{\partial r_i}{\partial q_j}\,\frac{\partial r_i}{\partial q_k}$$

The kinetic energy, T, is thus seen to be a homogeneous quadratic function of the *generalized velocities* or *motional* coordinates, $\dot{q}_j$, while the coefficients Q_{jk} are functions of the positional coordinates q_j. The potential energy, in general, will be a function of the positional coordinates.

We may now express Hamilton's principle in generalized coordinates; it becomes

$$\delta\int_{t_0}^{t_1} L\,dt = \int_{t_0}^{t_1}\sum_{j=1}^{s}\left[\frac{\partial L}{\partial \dot{q}_j}\delta\dot{q}_j + \frac{\partial L}{\partial q_j}\,\delta q_j\right]dt = 0$$

If we replace $\delta\dot{q}_j$ by $\dfrac{d(\delta q_j)}{dt}$, we find that the integral

$$\int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{q}_j}\delta\dot{q}_j\cdot dt = \int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{q}_j}\,d(\delta q_j)$$

$$= \frac{\partial L}{\partial \dot{q}_j}\delta q_j\bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{d}{dt}\,\frac{\partial L}{\partial \dot{q}_j}\delta q_j dt$$

where we have also integrated by parts But the generalized displacement, δq_j, at the instants t_0 and t_1, implies an arbitrary displacement of the ends of the paths, and since the initial and final positions of the system are fixed, the actual and generalized displacements vanish, so that the term $\left.\frac{\partial L}{\partial q_j}\delta q_j\right|_{t_0}^{t}$ is identically zero Hamilton's principle in generalized coordinates thus becomes

$$\int_{t_0}^{t_1}\left\{\sum_{j=1}^{s}\left(\frac{d}{dt}\frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial q_j}\right)\delta q_j\right\}dt = 0. \qquad (3\ 1)$$

Since the displacements δq_j are virtual but in general different from zero, the other factor of the integrand must vanish, or

$$\frac{d}{dt}\frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial q_j} = 0. \qquad (3.2)$$

This is one of Lagrange's equations of motion, in a holonomic system, a system in which the number of independent coordinates are just equal to the number of degrees of freedom, there is one such equation for each coordinate. In the above analysis we are considering a holonomic system with s degrees of freedom and s generalized coordinates; in the case of a free system of n particles s will of course be $3n$.

If some of the forces F_j' of the system are not conservative, then Hamilton's principle would read

$$\int_{t_0}^{t_1}(\delta L + \sum_{i=1}^{s} F_i' \ \delta r_i)dt = 0.$$

But since

$$\delta r_i = \sum_{j=1}^{s}\frac{\partial r_i}{\partial q_j}\delta q_j$$

and since the integral $\int_{t_0}^{t}\delta L dt$ will transform to generalized

coordinates into the integral already obtained, Lagrange's equations will have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j \tag{3 3}$$

where

$$Q_j = -\sum_{i=1}^{n} {}_iF' \frac{\partial r_i}{\partial q_j}$$

The Q_j so defined are generalized forces tending to change the coordinate q_j.

It very frequently happens that it is difficult to determine whether certain terms in the kinetic potential should be classed as kinetic or potential energy In fact, we found that the kinetic energy of a material system is a homogeneous quadratic function of the *motional* coordinates, whose coefficients were functions of the positional coordinates. Such terms may evidently be regarded as potential energy; but if any such term be so regarded its sign must be reversed, since the kinetic potential

$$L = T - W$$

must remain unchanged. This point is strongly emphasized by Livens,* and must be taken into account in a consistent theory of electric currents as a mechanical system.

4. The Electric Current as a Cyclic Mechanical System. We are naturally led through the electron hypothesis, the process of electrolysis, the discharge of electricity through gases and radiation in general to regard the energy of the electric current as of the kinetic type. It is customary to liken the electric current to a cyclic mechanical system In the cyclic systems studied by von Helmholtz, certain motional coordinates called cyclic coordinates entered the energy function while the corresponding positional coordinates were absent. The energy of a belt running around a pulley is determined by the velocity of one point

* See G H Livens' Treatise and Article, *loc cit*

of the belt, and this would evidently be independent of the positional coordinate, defining the position of the point on the belt

Thus if we consider two linear circuits with current strengths J_1 and J_2, the kinetic energy of the system will be a homogeneous quadratic function of the current strengths, or

$$T = \frac{1}{2c^2}L_1J_1{}^2 + \frac{M_{12}}{c^2}J_1J_2 + \frac{1}{2c^2}L_2J_2{}^2 \qquad (4.1)$$

We are here interpreting the current strength as a cyclic velocity The coefficients L_1, M_{12}, and L_2 will depend on the size, shape, and position of the conductors; the factor $\frac{1}{c_2}$, where c is the velocity of light, is introduced for comparison purposes later.

We may also find the energy of a system of electric currents by replacing each current circuit by its equivalent shell and then drop out the local part of the energy characterized by the intensity of magnetization of the magnetic shell. The potential energy of the equivalent shells, if there is no magnetic material in the field, will be given by the equation (II, 9 4)

$$W_m = -\frac{1}{8\pi}\int[B^2 - 16\pi^2I_0{}^2]d\tau \qquad (4.2)$$

where the integral is to be extended throughout all space. If we put the intensity of magnetization, I_0, equal to zero, we will have left the energy of the electric currents, since the energy of an electric current and the field energy of its equivalent shell are the same If we now regard this energy as of the kinetic type, its sign must be reversed, and we have for the kinetic energy of the system of currents

$$T = \frac{1}{8\pi}\int B^2d\tau. \qquad (4.3)$$

This of course is magnetic energy, and whether we call it kinetic or potential energy is immaterial if proper regard is

paid to sign. Because of its association with the electric current it is generally called *electrokinetic* energy If there is no condenser in the circuit and therefore no capacity, the energy of the electric field is negligible

To identify this last result (4.3) with the electrokinetic energy given by equation (4.1), and also to interpret the quantities L_1, L_2, and M_{12}, we use the vector potential A, already introduced and defined by equation

$$B = \text{curl } A. \tag{4.4}$$

For two steady currents imbedded in the ether $B = H$ and curl B is thus everywhere zero excepting in the current circuit, where, according to Maxwell's equations,

$$\text{curl } B = \frac{4\pi i}{c}$$

The vector B is thus to be determined from its vortices, or the vector potential A is to be determined from the vortices of the vector B. This problem is similar to the problem of determining a scalar potential from the sources of its gradient.

If we substitute from (4.4) into the above equation, we will find that

$$-\text{curl curl } A = -\frac{4\pi i}{c} \tag{4.5}$$

The vector A is somewhat arbitrary; it is restricted only by equation (4.5). If we impose the additional restriction that div A is identically zero, then since

$$\Delta' A = \text{grad div } A - \text{curl curl } A$$

equation (4.5) may be written in the form

$$\Delta' A = -\frac{4\pi i}{c}. \tag{4.6}$$

A special integral of this equation may be obtained by breaking it up into components (22) and solving three

scalar equations exactly similar to (I, 11 2) and synthesyzing, the result will evidently be

$$A = \frac{1}{c}\int \frac{i d\tau}{r}. \tag{4 7}$$

We will now replace B^2 in the kinetic energy given by equation (4.3) by B curl A, and transform the new integrand by means of formula (17*c*)

$$\operatorname{div} A \times B = B \cdot \operatorname{curl} A - A \operatorname{curl} B$$

This will give the equation

$$T = \frac{1}{8\pi}\int \operatorname{div} A \times B d\tau + \frac{1}{8\pi}\int A \cdot \operatorname{curl} B d\tau$$

for the electrokinetic energy

We can evaluate the first integral by considering the integral taken throughout a finite region, τ, bounded by a surface, σ, and then letting the surface σ recede to infinity in all directions Gauss's theorem transforms this proper integral into the surface integral, $\int A \times B \, n d\sigma$, taken over the bounding surface σ, which includes both circuits on its interior. From its definition (4 7) we see that the vector A vanishes at infinity like $\frac{1}{r}$; and since the vectors B and H are identical in free space, B vanishes at infinity like the gradient of a Newtonian potential function, so that the product, $A \times B$, vanishes at infinity like $\frac{1}{r^3}$ If a sphere be chosen as the surface σ, its area will become infinite like r^2, so that the integral in question vanishes like $\frac{1}{r}$, as r increases without limit.* We thus find the

* A more rigorous demonstration may be made by following the method used in a similar problem on magnetic field energy See Note (II, 8)

electrokinetic energy expressed by the equation

$$T = \frac{1}{8\pi}\int A \text{ curl } B d\tau = \frac{1}{2c}\int A \ i d\tau$$
$$= \frac{1}{2c^2}\int\int \frac{i\, d\tau \ i' d\tau'}{r}.$$

If we confine the triple integrals in the last form to the linear circuits, where i_1 and i_2 are different from zero, the two volume integrals will reduce to two line integrals both taken over the first circuit, both over the second, one over the first and the other over the second, and conversely, or if we write σ with its appropriate subscript or superscript for the area of the cross-section of the conductors, we will have

$$T = \frac{JJ'\sigma\sigma'}{8\pi c^2}\int\int \frac{dr \ dr'}{r}$$
$$= \frac{J_1{}^2\sigma_1{}^2}{8\pi c^2}\int\int\frac{dr_1 \ dr_1}{r} + \frac{2\sigma_1\sigma_2 J_1 J_2}{8\pi c^2}\int\int\frac{dr_1 \ dr_2}{r}$$
$$+ \frac{J_2{}^2\sigma_2{}^2}{8\pi c^2}\int\int\frac{dr_2 \ dr_2}{r}$$

since

$$i = Jdr \quad \text{and} \quad i' = J'dr'.$$

Upon identifying this result with equation (4 1) we find that

$$L_1 = \frac{\sigma_1{}^2}{4\pi}\int\int\frac{dr_1 \ dr_1}{r}, \quad L_2 = \frac{\sigma_2{}^2}{4\pi}\int\int\frac{dr_2 \ dr_2}{r}$$

and

$$M_{12} = M_{21} = \frac{\sigma_1\sigma_2}{4\pi}\int\int\frac{dr_1 \ dr_2}{r}$$

These coefficients evidently depend on the positional coordinates, namely, the size, shape and position of the circuits

For the interpretation of these coefficients we will find the electrokinetic energy, expressed in the form

$$T = \frac{1}{2c}\int A \ i d\tau$$

convenient We will replace $i d\tau$ by its equal, $J dr$, as before, and restrict the integration to the current circuits; the energy will evidently be given by the equation

$$T = \frac{J_1}{2c}\int_{0_1} A\, dr_1 + \frac{J_2}{2c}\int_{0_2} A\, dr_2$$

$$= \frac{J_1}{2c}\int_{\sigma_1} \text{curl } A \cdot n d\sigma_1 + \frac{J_2}{2c}\int_{\sigma_2} \text{curl } A \cdot n d\sigma_2$$

when we transform the last line integrals by Stokes' theorem; or

$$T = \frac{J_1}{2c}\int_{\sigma_1} B\, n d\sigma_1 + \frac{J_2}{2c}\int_{\sigma_2} B\, n d\sigma_2$$

where the surfaces σ_1 and σ_2 are arbitrary surfaces, having the conductors 1 and 2 for boundaries, respectively. If we write N_1 and N_2 for the total flux of induction through each of the two circuits, respectively, the above result takes the simpler form,

$$T = \frac{1}{2c}(J_1 N_1 + J_2 N_2)$$

If we compare this equation with equation (4.1) we will see that

$$N_1 = \frac{1}{c}(L_1 J_1 + M_{12} J_2)$$

and

$$N_2 = \frac{1}{c}(M_{21} J_1 + L_2 J_2).$$

But $\frac{J_1}{c}$ and $\frac{J_2}{c}$ are the current strengths in (c e m.) units; and since N_1 is the total flux of induction through circuit 1, we see that L_1 is the flux of induction due to current 1, per unit current through circuit 1; while M_{12} is the flux of induction due to current 2, per unit current through circuit 1 It is thus natural to call L_1, the coefficient of self-induction, and M_{12}, which is identical with M_{21}, the coefficient of mutual induction. Similarly L_2 is the coefficient self-induction for current 2

It is a simple matter to extend these results to n linear circuits It is sufficient to say that the electrokinetic energy in this case will have the form

$$T = \frac{1}{2c^2}[L_1J_1^2 + L_2J^2_2 + \quad . \\ + M_{12}J_1J_2 + \quad . .]$$

where the L's are the coefficients of self-induction and the M's are the coefficients of mutual induction. It is just as simple a problem to determine the values of the L's and M's and to prove that $M_{ij} = M_{ji}$, as in the case of two circuits.

5. The Magnetic Energy Due to Permanent Magnets and Current Circuits Combined.—In case the magnetic field is due both to electric currents and permanent magnets, we apply Ampère's conclusions regarding the magnetic shell and its equivalent circuit In the most general case where permanent magnetism, induced magnetism, and electric currents are involved, we can replace each current circuit by its equivalent shell and regard the shells as permanent magnets. After obtaining the total energy in proper form we can drop out the purely local part due to the shells, thus obtaining the energy of the original system

Equation (II, 9 4) is directly applicable for this purpose if we replace I_0 by $I_0 + I_s$, where I_s is the intensity of magnetization of the equivalent shells; equation (II, 9.4) then becomes

$$W_m = -\frac{1}{8\pi}\int[B^2 - 16\pi^2(I_0 + I_s + I)^2]d\tau + \tfrac{1}{2}\int I\cdot H d\tau$$

If we now replace the shells by their equivalent circuits the local part of the energy due to the shells drops out. This is effected by putting I_s equal to zero. We then have the total potential energy of the combined systems given by the equation

$$W_m = -\frac{1}{8\pi}\int[B^2 - 16\pi^2(I_0 + I)^2]d\tau + \tfrac{1}{2}\int I\cdot H d\tau.$$

This is identical in form with equation (II, 9.4), as might have been expected. We emphasized the fact that the ethereal energy of the magnetic field, if thought of as kinetic energy, is given by the equation

$$T = \frac{1}{8\pi}\int B^2 d\tau \tag{5.1}$$

and the kinetic energy density by $\frac{B^2}{8\pi}$.

6. The Dynamics of the Quasi-Stationary Current.—In the Maxwell theory of conduction currents it is assumed in general that the adjustment of the electromagnetic field is so rapid for energy change in the current or current circuit that the state at every instant may be regarded as stationary. We will restrict our discussion to cases covered by this hypothesis, or to the so-called quasi-stationary currents. This assumption implies that the displacement current is negligible in comparison with the conduction current, and that the time it takes an electromagnetic disturbance to travel from one part of the field to another is very short—relative to the time it takes to effect the corresponding change in the current or current circuit.

7. A Linear Circuit with Self-Inductance.—We will first study a single linear * current maintained by a constant impressed electromotive force E, and possessing a self-inductance L_1 and a resistance R. The kinetic energy of the current will be given by the equation

$$T = \frac{1}{c^2} L_1 J^2.$$

In the interpretation of the electric current as a mechanical system the electromotive force must be regarded as a mechanical force. The resistance in the circuit produces a counter e.m.f. equal in magnitude and opposite in sense to RJ which must be taken into account when writing the

* By a linear circuit we shall mean an isotropic wire or conductor, small but of definite finite cross-section, and that a linear relation holds between the current density I and the intensity E.

differential equation of the electric current. We also regard all magnetic energy as of the kinetic type, and electric energy stored up in a condenser as potential energy. Since

$$\frac{\partial T}{\partial J} = \frac{L_1}{c^2} J$$

Lagrange's equation of motion (3.3) for this mechanical system will be

$$\frac{L_1}{c^2} \frac{dJ}{dt} = E - RJ$$

or

$$\frac{L_1}{c^2} \frac{dJ}{dt} + RJ = E.$$

This is a very simple linear differential equation; it may be solved in various ways. Its general solution is

$$J = c_1 e^{\frac{-c^2Rt}{L_1}} + \frac{E}{R}$$

where c_1 is the arbitrary constant of integration. This constant must be determined from the initial conditions. If when t is zero, the current, J, is zero, then c_1 will be equal to $-\frac{E}{R}$, or the particular solution here sought is

$$J = \frac{E}{R}(1 - e^{\frac{-c^2Rt}{L_1}}).$$

The initial conditions are somewhat ideal; in fact, during the establishment of the current the changes are so rapid that the state can not be considered as stationary. The condition here implies an instantaneous establishment of the current strength. However, the duration for this is so brief that the situation is handled as an impulse, i.e., ignored unless the impulse is a special subject of investigation

Another simple example, omitted here, which may be treated in a similar manner, is the case in which the

impressed electromotive force is periodic, possibly defined by the equation

$$E = - \sin \omega t.$$

This periodic e m f may be caused by the swinging of a magnet or by some other device

8 **A Linear Circuit with an Impressed E.M.F. Capacity and Resistance.**—We will consider a single linear circuit, and if to the conditions imposed in the previous section we add the condition that there is a condenser in the circuit, then the electrokinetic energy of the system will remain unchanged. It will be given as in the previous case by the equation

$$T = \frac{1}{2c^2} L_1 J^2.$$

But the Lagrangian function, L, will now involve the potential energy of the condenser We have already found that the mutual potential energy of a system is $\frac{1}{2}\Sigma q\phi$, so the potential energy, W, of the condenser will be

$$W = \tfrac{1}{2}\int_\tau (\phi_2 - \phi_1)\rho d\tau + \tfrac{1}{2}\int_\sigma (\phi_2 - \phi_1)\omega d\sigma.$$

But since the electric volume density is zero, this reduces to the form

$$W = \tfrac{1}{2}(\phi_2 - \phi_1)\int \omega d\sigma = \tfrac{1}{2}(\phi_2 - \phi_1)q$$

if q is the charge on the positive plate. Upon introducing the capacity of the condenser

$$C = \frac{q}{\phi_2 - \phi_1}$$

we find that

$$W = \frac{C}{2}(\phi_2 - \phi_1)^2.$$

Or if we assume the negative plate to be the position of zero potential, so that $\phi_1 = 0$, we will have the final form

$$W = \frac{C}{2}\phi^2$$

where ϕ written for ϕ_2 is the difference of potential or the potential at the positive plate. The Lagrangian function thus takes the form

$$L = \frac{L_1 J^2}{2c^2} - \tfrac{1}{2} C\phi^2$$

in this case.

The impressed force here will be the difference of potential between the condenser plates and the counter-electromotive force of resistance, RJ. Again the kinetic-potential involves only the cyclic velocity, J, so that the system is defined by the single Lagrangian equation

$$\frac{L_1}{c^2} \frac{dJ}{dt} = \phi - RJ \qquad (8\ 1)$$

obtained by substituting the kinetic potential in equation (3.3), and replacing Q by the electromotive force, $\phi - RJ$

Since now the capacity $C = \dfrac{q}{\phi}$ we may express the current strength, J, in the form

$$J = -\frac{dq}{dt} = -C\frac{d\phi}{dt}$$

So if we replace J by $-C\dfrac{d\phi}{dt}$ in equation (8.1) we shall find that ϕ satisfies the homogeneous differential equation

$$\frac{L_1 C}{c^2}\frac{d^2\phi}{dt^2} + CR\frac{d\phi}{dt} + \phi = 0 \qquad (8\ 2)$$

If on the other hand we take the time derivative of each member of equation (8 1), we find that

$$\frac{L_1}{c^2}\frac{d^2J}{dt^2} = \frac{d\phi}{dt} - R\frac{dJ}{dt};$$

and upon replacing $\dfrac{d\phi}{dt}$ by its equal, $-\dfrac{J}{C}$, we obtain the defining equation for J in the form

$$\frac{L_1 C}{c^2}\frac{d^2J}{dt^2} + CR\frac{dJ}{dt} + J = 0 \qquad (8\ 3)$$

We thus see that the current J and the potential ϕ satisfy one and the same differential equation

Since equation (8.2) is linear and homogeneous, we may obtain its solution by assuming that

$$\phi = e^{\lambda t}$$

as in the previous case. A substitution back into equation (8.2) will show that this is a solution if λ satisfies the equation

$$\frac{L_1C}{c^2}\lambda^2 + CR\lambda + 1 = 0$$

or

$$\lambda = \frac{-c^2R}{2L_1} \pm \frac{c^2\sqrt{R^2 - \frac{4L_1}{c^2C}}}{2L_1}$$

There are two cases to be considered, one arising when

$$R^2 \geq \frac{4L_1}{c^2C}$$

and when

$$R^2 < \frac{4L_1}{c^2C}$$

Case I: $R^2 \geq \dfrac{4L_1}{c^2C}$—In this case the roots of the characteristic equation, which we will designate by λ_1 and λ_2, are both real and negative. In both cases our solution, in this notation, will be given by the equation

$$\phi = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}. \tag{8.2}$$

And since

$$J = -C\frac{d\phi}{dt}$$

$$J = -C(c_1\lambda_1e^{\lambda_1 t} + c_2\lambda_2e^{\lambda_2 t}). \tag{8.3}$$

The arbitrary constants of integration c_1 and c_2 are to be determined from the initial conditions For these we will assume that when t is zero the current is zero, and the charge on the condenser plate is q_0. If we substitute these conditions into equation (8.3) we find the relation

$$c_1\lambda_1 + c_2\lambda_2 = 0 \tag{8.4}$$

between the arbitrary constants. When we substitute zero for t in equation (8.2), we obtain the potential at that instant, or

$$\phi = c_1 + c_2$$

so that the capacity

$$C = \frac{q_0}{c_1 + c_2}$$

This is the second condition on the arbitrary constants; it may be written in the form

$$c_1 + c_2 = \frac{q_0}{C}. \tag{8.5}$$

When we solve (8.4) and (8.5) for c_1 and c_2 we find that

$$Cc_1 = \frac{q_0\lambda_2}{\lambda_2 - \lambda_1}, \quad Cc_2 = -\frac{q_0\lambda_1}{\lambda_2 - \lambda_1}$$

We thus have

$$C\phi = \frac{q_0}{\lambda_2 - \lambda_1}(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t})$$

and

$$J = -\frac{q_0\lambda_1\lambda_2}{\lambda_2 - \lambda_1}(e^{\lambda_1 t} - e^{\lambda_2 t})$$

for our particular solutions.

Since the roots λ_1 and λ_2 are each negative in this case, even if they are repeated roots, the potential, ϕ, and the current, J, each approach zero as the time increases This is the situation after the currents have been established. There is thus a gradual discharge of the condenser and a dying out of the current. The discharge is non-periodic, while the current is direct or non-oscillatory

Case II: $R^2 < \dfrac{4L_1}{c^2C}$, *Oscillatory Discharge.*—If we define the quantities a and b by the equations

$$a = \frac{c_2R}{2L_1}, \quad b = \frac{c^2}{2L_1}\sqrt{\frac{4L_1}{c^2C} - R^2}$$

then the solution of (8 2) may be written in the form

$$\phi = e^{-at}(c_1 \cos bt + c_2 \sin bt) \tag{8.6}$$

and since

$$J = -C\frac{d\phi}{dt}$$

$$J = Ce^{-at}[(c_1a - c_2b) \cos bt + (c_1b + c_2a) \sin bt] \tag{8.7}$$

If we assume again that when t is zero the current, J, is zero, and the charge on the condenser plate is q_0, and substitute these initial conditions into equation (8 7), we shall find that

$$bc_2 - c_1a = 0$$

Upon substituting zero for t in equation (8 6), we find the initial value of ϕ to be the constant c_1. The capacity of the condenser is thus given by the equation

$$C = \frac{q_0}{c_1}$$

or

$$c_1 = \frac{q_0}{C}$$

while the other integration constant

$$c_2 = \frac{aq_0}{bC}.$$

If we substitute these values of the arbitrary constants into equations (8 6) and (8.7), we shall have

$$\phi = \frac{q_0e^{-at}}{bC}(b \cos bt + a \sin bt)$$

and

$$J = \frac{q_0}{b}(a^2 + b^2)\ e^{-at} \sin bt$$

for the particular solutions.

The voltage and the current strength are thus seen to be periodic in this case, of period

$$\frac{2\pi}{b} \equiv \frac{4\pi L_1}{c^2\sqrt{\frac{4L_1}{c^2C} - R^2}}$$

then the solution of (8 2) may be written in the form

$$\phi = e^{-at}(c_1 \cos bt + c_2 \sin bt) \tag{8.6}$$

and since

$$J = - C\frac{d\phi}{dt}$$

$$J = Ce^{-at}[(c_1a - c_2b) \cos bt + (c_1b + c_2a) \sin bt] \tag{8.7}$$

If we assume again that when t is zero the current, J, is zero, and the charge on the condenser plate is q_0, and substitute these initial conditions into equation (8 7), we shall find that

$$bc_2 - c_1a = 0$$

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$$C = \frac{q_0}{c_1}$$

or

$$c_1 = \frac{q_0}{C}$$

while the other integration constant

$$c_2 = \frac{aq_0}{bC}.$$

If we substitute these values of the arbitrary constants into equations (8 6) and (8 7), we shall have

$$\phi = \frac{q_0e^{-at}}{bC}(b \cos bt + a \sin bt)$$

and

$$J = \frac{q_0}{b}(a^2 + b^2)\ e^{-at} \sin bt$$

for the particular solutions.

The voltage and the current strength are thus seen to be periodic in this case, of period

$$\frac{2\pi}{b} = \frac{4\pi L_1}{c^2\sqrt{\frac{4L_1}{c^2C} - R^2}}$$

We have here an oscillatory discharge of the condenser, since the amplitude decreases with the time The sense of the current and of the potential difference are periodically reversed, in the phenomenon.

For the particular case where R^2 is small and negligible, the period becomes

$$\frac{2\pi}{b} = \frac{2\pi}{c}\sqrt{CL_1}$$

The situation here will not be essentially different from the more general case treated.

Another asymptotic case is obtained by putting the capacity equal to infinity. This condition makes b zero, so that the solution degenerates into that obtained in the previous section.

The reason for the discharge of the condenser in both cases treated is apparent; the current produces a counter induced electromotive force of self-induction. In *Case I*, the resistance is so great that the condenser is discharged without a reversal of the current sense. In *Case II*, the resistance is small, and the counter e.m.f actually reverses the current sense many times, theoretically an infinite number, as the amplitude approaches zero.

Other cases of interest and of the same general type are cases involving a constant, or a periodic, impressed e.m.f. in addition to the e.m.f. assumed. Then for two currents the kinetic potential is

$$T = \frac{1}{2c^2}[L_1J_1^2 + 2M_{12}J_1J_2 + L_2J_2^2]$$

If the circuits each contain a fall of potential and a resistance ϕ_1 and R_1 for the first and ϕ_2 and R_2 for the second, the Lagrangian equations will be

$$\frac{1}{c^2}L_1\frac{dJ_1}{dt} + \frac{1}{c^2}M_{12}\frac{dJ_2}{dt} = \phi_1 - J_1R_1$$

$$\frac{1}{c^2}L_2\frac{dJ_2}{dt} + \frac{1}{c^2}M_{12}\frac{dJ_1}{dt} = \phi_2 - J_2R_2.$$

The discussion of the solution of these equations will be omitted. Sufficient has been said to indicate the dynamical interpretation of the electric current, and the generalization to cases of more than two circuits should present no difficulty.

EXERCISES

(1) For a system of free particles is D'Alembert's principle true? If so, of what use is it? What is the general situation wherein D'Alembert's principle is important and useful? Just what does it do for the equations of motion?

(2) Write out the Lagrangian function and deduce the equations of motion from it for the simple cases.

> (*a*) a body falling near the earth's surface; (*b*) a case of simple harmonic motion, (*c*) the simple pendulum, using the angle between the bob and the vertical as the generalized coordinate.

(3) Write out the Lagrangian function and the equations of motion for a particle constrained to move on a perfectly smooth sphere.

(4) Analyze the single linear current maintained by a decadent impressed e.m.f., $E = e^{-\lambda t}$, with a self-inductance L_1 and a resistance R.

(5) Carry out the solution of the problem suggested towards the end of Art. 8.

(6) Write out the total energy density in free space for an electromagnetic field

(7) Just what difficulties arise when one tries to write out the equations of motion of a single electron?

(8) Solve the problem of a linear circuit having capacity, resistance, and the additional decadent e.m.f , $e^{-\lambda t}$.

CHAPTER V

THE ELECTRON THEORY

1. In Chapter IV we left open the question of the applicability of the Maxwell theory We found, however, in the case considered, and this is the situation in general according to the Maxwell theory, that electromagnetic disburbances are propagated through material media with the velocity of propagation equal to $\frac{c}{\sqrt{\epsilon\mu}}$. If the disturbance is propagated through a rare gas or if it is a slowly changing disturbance there is a close agreement between theory and experiment, but for rapidly changing electromagnetic disturbances propagated through ordinary material media the theory seriously breaks down

In fact if these disturbances are light waves we see that, according to the theory, the velocity of propagation is independent of the wave-length and depends only on the constants that characterize the medium; while experiment shows that light of different wave-lengths is propagated through material media with different velocities Theory also predicts that, if a medium is brought into an electric or a magnetic field, the field will have no effect on the propagation of an electromagnetic disturbance through the medium. This is not at all true, as is instanced by the Stark and Zeeman effects in the splitting of the spectral lines under the influence of electric or magnetic fields

The disagreement between the Maxwell theory and experiment is due, in part, to the restrictions imposed by the constants ϵ, μ, and κ. The removal of the restrictions imposed by these constants leaves us free to speculate on

the underlying mechanism that is the primary cause of all electromagnetic phenomena

The electron theory as developed by Larmor and H A. Lorentz rids the theory of these constants altogether, basing it on the atomistic structure of electricity. We have already seen how the electron hypothesis, the assumption of ultimate nuclear charges, furnishes an explanation of many electric phenomena In fact this assumption is absolutely confirmed by the α and β rays and the isolation and measurement of the electronic charge. The purpose of the electron theory is to explain all electromagnetic phenomena by the distribution and motion of such charges. The objects of consideration are these electric corpuscles (the protrons and electrons) and the ether.

2. The Fundamental Equations for the Electron Theory.—We will adopt the Lorentz point of view and regard these corpuscles, whether proton or electron, as a modification of the ether, certain points of the ether will then be characterized by a density ρ, different from zero. We will assume that the corpuscle has finite dimensions, and that at the surface there is a transition layer where the density changes rapidly and continuously from zero to a finite value This makes the density ρ, which is zero for the ether, a continuous point function. For some purposes we may also, if we wish, regard the corpuscle as a point charge, an idea exactly similar to the mass point so useful in mechanics.

We are dealing then with electric charges embedded in the ether Thus according to Gauss's electric flux theorem the density, ρ, of such a charge multiplied by 4π, is the divergence of the electric intensity The total current under these conditions will be a displacement current in the ether and a convection current due to the motion of the corpuscles In fact, we have already seen how the conduction current might be regarded as a convection current due to the motion of electrons through material media. Then Rowland's experiments show that the con-

vection current produces a magnetic field not different from that produced by the conduction current

Since in free space the Maxwell theory agrees well with experiment, we assume that the Maxwell equations (III, 14 1) for stationary media are valid for the ether where the permeability, μ, and the specific inductive capacity, ϵ, are each unity If we use the small letters, $\boldsymbol{e}$ and $\boldsymbol{h}$, for the electric and magnetic intensity, respectively, for this case, and regard the conduction current, $\boldsymbol{\imath}$, as a convection current, $\rho\boldsymbol{u}$, these equations may be written in the following form

$$\left.\begin{aligned} \frac{1}{c}\left(4\pi\rho\boldsymbol{u} + \frac{\partial \boldsymbol{e}}{\partial t}\right) &= \operatorname{rot} \boldsymbol{h} \\ -\frac{1}{c}\frac{\partial \boldsymbol{h}}{\partial t} &= \operatorname{rot} \boldsymbol{e} \\ \rho &= \frac{1}{4\pi}\operatorname{div} \boldsymbol{e} \\ 0 &= \operatorname{div} \boldsymbol{h} \end{aligned}\right\} \qquad (2\ 1)$$

These are the fundamental field equations of the electron theory, as deduced by H A Lorentz, for electric corpuscles embedded in and moving through the ether However, the hypothesis of ultimate nuclear charges is not involved in this set of equations. It is an additional hypothesis used in completing the set of defining equations.

3. The Conservation of Charge and the Flow of Energy.—From the set of field equations (2 1), we are able to deduce two characteristic properties First, the conservation of electric charge is obtained from the first and third of these equations. If we take the divergence of each member of the first equation of the set (2.1), we will have the continuity condition

$$\frac{1}{4\pi}\frac{\partial \operatorname{div} \boldsymbol{e}}{\partial t} + \operatorname{div} \rho\boldsymbol{u} = 0$$

as a result. If we replace div $\boldsymbol{e}$ by $4\pi\rho$, from the third of

(2.1), and integrate throughout the volume τ, bounded by a surface σ, we obtain the equation

$$\left.\begin{aligned} -\frac{\partial}{\partial t}\int_\tau \rho d\tau &= \int_\tau \operatorname{div} \rho u d\tau \\ \text{or} \qquad -\frac{\partial}{\partial t}\int_\tau \rho d\tau &= \int_\sigma \rho u \; n d\sigma \end{aligned}\right\} \tag{3.1}$$

when we transform the right member by Gauss's theorem.

This last equation states the fact that *the time rate of decrease of electric charge in the volume, τ, is just equal to the flux of electricity through the bounding surface.* Electricity is thus transferred from one region to another, but never created or destroyed.

The second theorem characterizes the flux of energy through a surface σ, bounding a region τ. We obtain the desired result by first substituting the values of curl e and curl h in the identity (17c)

$$\operatorname{div} e \times h = h \cdot \operatorname{curl} e - e \operatorname{curl} h$$

from the first two equations of the set (2.1) This identity then becomes

$$\operatorname{div} e \times h = -\frac{1}{c}(e \cdot \dot{e} + h \cdot \dot{h}) + \frac{4\pi\rho}{c} e \cdot u$$

or

$$\operatorname{div} \frac{c}{4\pi} e \times h = -\frac{1}{8\pi}\frac{\partial}{\partial t}(e^2 + h^2) - \rho e \cdot \frac{dr}{dt}.$$

If we integrate this last identity throughout an arbitrary volume, τ, we will have the equation

$$-\frac{\partial}{\partial t}\int_\tau \frac{1}{8\pi}(e^2 + h^2)d\tau - \int_\tau \rho e \cdot \frac{dr}{dt} d\tau = \int_\tau \operatorname{div} \frac{c}{4\pi} e \times h d\tau.$$

Or if we replace the vector $\frac{c}{4\pi} e \times h$, called the *Poynting vector* or the *vector radiant*, by the letter s, and transform

the right member of the last equation by Gauss's theorem, we obtain the equation

$$-\frac{\partial}{\partial t}\int_{\tau}\frac{1}{8\pi}(e^2 + h^2)d\tau - \int_{\tau}\rho e\ \frac{dr}{dt}d\tau = \int_{\sigma}s\ \ nd\sigma \quad (3.2)$$

for our final form

This result, known as Poynting's theorem, is easily interpreted The first integral is evidently the total field energy of the electromagnetic field situated in the region τ. In the second integral, the factor $\rho e d\tau$ is force. This factor scalarly multiplied by $\frac{dr}{dt}$ is work per unit time, or the whole integral represents the time rate of decrease of the kinetic energy of the corpuscles in the volume τ. We may thus state Poynting's theorem as follows

The time rate of decrease of energy in the volume τ is equal to the flux of the vector radiant through its bounding surface σ, or the efflux of energy through the surface σ is just equal to the flux of the vector radiant through this surface. In addition to the distribution of electromagnetic energy throughout the ether, we have here then the flow of energy from volume element to volume element.

4. The Hypothesis of the Ultimate Nuclear Charge.—The set of field equations (2.1), though fundamental, do not constitute a complete set of equations for the determination of the vectors u, e, and h, and the density, ρ, which characterizes the distribution of charge. If we speak in terms of the coordinates of the three vectors, we may say that we have ten unknowns and but eight defining equations; we are thus left free to make some additional hypothesis governing the motion and distribution of the charges. This is done through the assumption that electricity exists in ultimate nuclear charges, and that these corpuscles or elementary charges are susceptible to a mechanical force. This implies that the protons and electrons are substantial and possess the property of inertia inherent in ponderable matter.

It is assumed that the total mechanical force of electromagnetic origin acting on a corpuscle is made up of two components. The first of these components per unit charge is the electric field intensity, e, or the force components acting on the element of charge, $\rho d\tau$, is $\rho e d\tau$. The second component is easily obtained by identifying it with the electric intensity induced in a conductor when the conductor is moved through a magnetic field.

In obtaining Faraday's law, as expressed by equation (III, 12.3), we regarded the conductor as fixed in position while the magnetic field was made to change with the time. If we regard the circuit as moving through the magnetic field, equation (III, 12.3) must be written in the general form

$$\int_\sigma \operatorname{curl} E \cdot n d\sigma = -\frac{1}{c}\frac{\partial}{\partial t}\int_\sigma B \cdot n d\sigma \qquad (4.1)$$

since the differentiation under the integral sign is no longer admissible.

The right member of this equation is evidently made up of two parts, one, the time rate of change of flux through the fixed circuit due to the time variation of the induction and a second contribution, the time rate of change of flux due to the motion of the circuit through the magnetic field regarded as stationary. Equation (4.1) may then be written as follows

$$\int_\sigma \operatorname{curl} E \cdot n d\sigma = -\frac{1}{c}\int_\sigma \frac{\partial B}{\partial t} n d\sigma - \frac{1}{c}\frac{\partial}{\partial t}\int_\sigma B \cdot n d\sigma \qquad (4.2)$$

where by the second term in the right member we mean the time rate of decrease of flux of induction through the circuit due to the motion of the conductor through the fixed magnetic field.

This second term may be evaluated by considering the total flux of induction through the closed surface composed of the initial and final positions of the surface, σ, at the beginning and end of the time interval dt, together with the cylindrical surface generated by the conductor during this

time interval. Since this is a closed surface and div B is identically zero, Gauss's theorem furnishes us the fact that the total flux of induction through this surface is zero.

The flux through an element of the lateral surface is $dtdr \times v\ B$ For if dr is an element of the circuit and v its velocity, vdt will be the distance the element of circuit has moved, and the vector product $dtdr \times v$ will not only be the area of the elementary parallelogram swept out but it is the outer normal to the cylindrical surface. Thus the flux through the lateral surface will be given by the integral $dt\int_s dr \times v \cdot B$. The total flux then takes the form

$$\int_{\sigma'} B\ nd\sigma - \int_{\sigma} B\ nd\sigma + dt\int_s dr \times v\ B = 0$$

where we are using σ' to denote the final position of the surface σ. This last equation may be written in the form

$$\frac{1}{dt}\left(\int_{\sigma'} B \cdot nd\sigma - \int_{\sigma} B\ nd\sigma\right) = -\int_s v\ B \times dr$$

thus in the limit the time rate of change of flux of induction through the surface σ is given by the line integral $-\int_s v \times B \cdot dr$, taken around the conducting circuit. We can transform this integral by Gauss's theorem into the surface integral, $-\int_\sigma \text{curl}\,(v \times B) \cdot nd\sigma$, then, by simply substituting in equation (4.2) this equation will assume the following form.

$$\int_\sigma \text{curl}\, E\ nd\sigma = -\frac{1}{c}\int_\sigma \frac{\partial B}{\partial t} \cdot nd\sigma + \frac{1}{c}\int_\sigma \text{curl}\,(v \times B)\ nd\sigma.$$

For a conductor moving through a stationary magnetic field the first integral in the right member vanishes, and we have the relation

$$\int_\sigma \text{curl}\, E \cdot nd\sigma = \frac{1}{c}\int_\sigma \text{curl}\,(v \times B) \cdot nd\sigma$$

and since the surface σ is arbitrary

$$\operatorname{curl} E = \operatorname{curl} \frac{1}{c}(v \times B).$$

The general solution of this differential equation is

$$E = \frac{1}{c} v \times B + \operatorname{grad} \phi$$

where ϕ is an arbitrary scalar point function. But when the induction is zero the induced electric intensity is zero; hence grad ϕ is identically zero and our solution is simply

$$E = \frac{1}{c} v \times B$$

or the intensity induced in the conductor is perpendicular to both the magnetic field and the direction of motion.

This result and the experiments of Rowland justify us in assuming that a stream of corpuscles traveling through a magnetic field in free space of magnetic intensity h, and with a velocity u, is acted on by a force characterized by the intensity $\frac{1}{c} u \times h$, the induction and magnetic intensity being identical for the ether. The total external force, then, of electromagnetic origin, acting on an element of charge, $\rho d\tau$, is

$$f = \rho d\tau\left(e + \frac{1}{c} u \times h\right). \tag{4.3}$$

In the light of the electrostatic theory where like charges repel like charges with a force following the Newtonian law, it would appear that, if all the forces acting on the element of charge, $\rho d\tau$, were of electromagnetic origin, the electric corpuscle would be disrupted. However, we are led to the assumption that the resultant force of electric origin acting on an electric corpuscle is given by the integral

$$F = \int \rho d\tau\left(e + \frac{1}{c} u \times h\right) \tag{4.4}$$

taken throughout the volume of the corpuscle. This implies that all internal forces of electromagnetic origin tending to disrupt the corpuscle are in equilibrium with some mechanical forces, and the corpuscle thus persists as an entity. As to the nature of these mechanical forces little may be said But if all forces are of electromagnetic origin, then just how the corpuscle is maintained in its entirety is still more obscure

5. The Electromagnetic Potentials.—The fundamental problem of electrodynamics is to find the distribution and motion of the charges for all time, having given the state of the electromagnetic field at a given instant. The field equations (2.1) were found to be inadequate for this purpose, so we are seeking a set of equations that will do this. If we add to the set (2.1), the equations of motion of the electric corpuscles, we may then regard the set as complete

In the less general but very important problem of finding the state of the field when the distribution and motion of the charges are prescribed, the field intensities are determinate and expressible explicitly in terms of certain potential functions termed delayed potentials. These potential functions are deducible directly from the field equations (2 1), and are thus pertinent even in the more general problem. The potentials, we shall introduce here, though closely related to the scalar and vector potentials used before, will be defined independently—without reference to those previously used but with regard to their applicability in determining the intensities $\boldsymbol{e}$ and $\boldsymbol{h}$, defined by the field equations.

The fourth equation of the set (2.1) defines the magnetic intensity, $\boldsymbol{h}$, as a solenoidal vector; it is thus the curl of a second vector, $\boldsymbol{a}$, so that we may write

$$\boldsymbol{h} = \operatorname{curl} \boldsymbol{a}. \tag{5.1}$$

Further restrictions will be placed on the vector $\boldsymbol{a}$ later. This is permissible, since all we wish is to introduce some auxiliary functions in terms of which the field intensities

may be expressed. If we substitute the relation (5.1) into the second of (2.1), we will find that

$$\operatorname{curl}\left(e + \frac{1}{c}a\right) = 0.$$

But this equation implies that the argument $e + \frac{1}{c}a$ is a lamellar or potential vector. If we designate the potential by ϕ we may write the equation

$$e + \frac{1}{c}a = -\operatorname{grad}\phi$$

or

$$e = -\frac{1}{c}a - \operatorname{grad}\phi \tag{5.2}$$

The scalar ϕ and the vector a are the potential functions referred to; as soon as these functions are known the intensities e and h are obtainable according to the manner indicated in equations (5.1) and (5.2).

The fact that the vectors e and h also satisfy the first and third of the set (2.1), imposes additional restrictions on the potential functions. Thus if we substitute from (5.2) into the third of (2.1) we will have

$$\operatorname{div} e = -\operatorname{div}\frac{1}{c}a - \Delta\phi = 4\pi\rho \tag{5.3}$$

We restricted the vector potential, in our previous treatment of it, by assuming that it was a solenoidal vector, or that div a was zero; we here make the simplifying assumption that

$$\operatorname{div} a = -\frac{\phi}{c} \tag{5.4}$$

which relates the two potentials. If we now substitute this last condition into (5.3) we will have the differential equation,

$$\Delta\phi - \frac{1}{c^2}\phi = -4\pi\rho, \tag{5.5}$$

defining the scalar potential ϕ.

We next substitute curl a for h from (5 1) into the first of (2 1) and find that

$$-\text{curl curl } a = -\frac{4\pi\rho}{c} u - \frac{1}{c} \dot{e}$$

But if we replace $-$curl curl a by its equal, $\Delta' a$ $-$ grad div a (20) and e by $-\frac{1}{c}\dot{a} -$ grad ϕ from (5 2) this last equation will take the following form

$$\Delta' a - \text{grad div } a = -\frac{1}{c} 4\pi\rho u + \frac{1}{c^2}\ddot{a} + \text{grad}\frac{1}{c}\dot{\phi}$$

Also since the potentials are related through equation (5.4) the gradients just cancel each other and this simplifies into the equation

$$\Delta' a - \frac{1}{c^2}\ddot{a} = -\frac{1}{c} 4\pi\rho u. \tag{5.6}$$

We thus see that the potentials are defined by equations (5 5) and (5.6) in terms of the distribution and motions of the charges supposed to be known and characterized by the density ρ and the velocity u. The electromagnetic field is then given in terms of these potentials through the equations (5.1) and (5 2).

6. Electromagnetic Waves.—In the ether where the electric density ρ is zero, the electromagnetic potentials defined by equations (5.5) and (5 6), take the simplified form

$$\left.\begin{aligned} \Delta\phi &= \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \\ \Delta' a &= \frac{1}{c^2}\frac{\partial^2 a}{\partial t^2} \end{aligned}\right\} \tag{6.1}$$

when ρ is placed equal to zero in these equations. We shall see that these are typical wave equations, so we are led to the conclusion that all electromagnetic disturbances are propagated through space in the form of waves

If the character of a disturbance set up in the ether and propagated in all directions is independent of the direction,

the wave form would be spherical and a function of the time and the distance from the source of the disturbance. If we thus assume that the potentials ϕ and $\mathbf{a}$ are functions of r and t alone, equations (6.1) may be put into forms which are easily solved. These solutions will exhibit in a general way the character of electromagnetic disturbances traveling through space. We will have then, using the transformation theorem (17*a*) and the definition for the operator Δ,

$$\Delta\phi = \operatorname{div}\operatorname{grad}\phi = \operatorname{div}\left(\frac{\partial\phi}{\partial r}\operatorname{grad} r\right)$$

$$= \frac{\partial\phi}{\partial r}\operatorname{div}\operatorname{grad} r + \operatorname{grad} r \cdot \operatorname{grad}\frac{\partial\phi}{\partial r}$$

$$= \frac{\partial\phi}{\partial r}\operatorname{div}\operatorname{grad} r + \frac{\partial^2\phi}{\partial r^2}\operatorname{grad} r \cdot \operatorname{grad} r$$

But since

$$\operatorname{div}\operatorname{grad} r = \frac{2}{r}$$

and grad r is the unit vector, $\frac{1}{r}\mathbf{r}$, this last expression takes the simple form

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}$$

which may evidently be written as

$$\frac{1}{r}\frac{\partial^2 r\phi}{\partial r^2}$$

It will also be quite evident that $\Delta'\mathbf{a}$ will have the form

$$\frac{1}{r}\frac{\partial^2 r\mathbf{a}}{\partial r^2}\text{*}$$

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* $\Delta'\mathbf{a} = (\Delta a_x)\mathbf{i} + (\Delta a_y)\mathbf{j} + (\Delta a_z)\mathbf{k}$

$$= \frac{1}{r}\left[\frac{\partial^2 ra_x}{\partial r^2}\mathbf{i} + \frac{\partial^2 ra_y}{\partial r^2}\mathbf{j} + \frac{\partial^2 ra_z}{\partial r^2}\mathbf{k}\right] = \frac{1}{r}\frac{\partial^2 r\mathbf{a}}{\partial r^2}$$

so that equations (6 1) reduce to

$$\left.\begin{aligned}\frac{\partial^2 r\phi}{\partial r^2} &= \frac{1}{c^2}\frac{\partial^2 r\phi}{\partial t^2}\\ \frac{\partial^2 ra}{\partial r^2} &= \frac{1}{c^2}\frac{\partial^2 ra}{\partial t^2}\end{aligned}\right\} \tag{6.2}$$

Now these equations in $r\phi$ and ra as dependent variables are identical in form with the wave equations already considered, so their solutions may be written down by analogy; we thus have

$$\left.\begin{aligned} r\phi &= \phi_1\left(t - \frac{r}{c}\right) + \phi_2\left(t + \frac{r}{c}\right)\\ ra &= a_1\left(t - \frac{r}{c}\right) + a_2\left(t + \frac{r}{c}\right)\end{aligned}\right.$$

or explicitly

$$\left.\begin{aligned} \phi &= \frac{1}{r}\left[\phi_1\left(t - \frac{r}{c}\right) + \phi_2\left(t + \frac{r}{c}\right)\right]\\ a &= \frac{1}{r}\left[a_1\left(t - \frac{r}{c}\right) + a_2\left(t + \frac{r}{c}\right)\right].\end{aligned}\right\} \tag{6.3}$$

These are in a sense the general solutions, since ϕ and a involve arbitrary functions of the arguments indicated. The velocity of propagation here is c, the velocity of light. The case considered in Chapter III where the velocity of propagation is $\frac{c}{\sqrt{\epsilon\mu}}$ could have been treated in the same way.

For the wave forms traveling out from the source of the disturbance we need write

$$\left.\begin{aligned}\phi &= \frac{1}{r}\phi_1\left(t - \frac{r}{c}\right)\\ a &= \frac{1}{r}a_1\left(t - \frac{r}{c}\right).\end{aligned}\right\} \tag{6.4}$$

The electromagnetic intensities obtained by substituting these values in equations (5 1) and (5.2) are thus

$$\left.\begin{aligned} e &= -\frac{1}{cr}\,a'_1\left(t - \frac{r}{c}\right) - \operatorname{grad}\frac{1}{r}\,\phi_1\left(t - \frac{r}{c}\right) \\ h &= \operatorname{curl}\frac{1}{r}\,a_1\left(t - \frac{r}{c}\right) \end{aligned}\right\} \quad (6\ 5)$$

where the superscript here used indicates the derivative of a_1 with respect to the argument $t - \frac{r}{c}$.

7. Plane Waves—The waves considered in the last section were spherical in form. If the wave front is far removed from the source of the disturbance it may be regarded as a plane wave We proceed to apply the previous theory to this situation.

If we regard the wave as perpendicular to the axis OX, and traveling outward from the origin of coordinates along this axis, the electrodynamic potentials will be functions of x and t alone. Equations (6 1) defining these potentials then simplify into the equations

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\frac{\partial^2 a}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 a}{\partial t^2}.$$

We are thus justified in assuming that the potentials have the form

$$\left.\begin{aligned} \phi &= \phi\left(t - \frac{x}{c}\right) \\ a &= a\left(t - \frac{x}{c}\right) \end{aligned}\right\} \quad (7.1)$$

for the disturbance traveling outward from the origin, the source of the disturbance.

But these two functions are related through equation (5.4), which relation in this case becomes

$$\frac{\partial a_x}{\partial x} = -\frac{1}{c}\frac{\partial \phi}{\partial t} \tag{7.2}$$

Also

$$\frac{\partial a_x}{\partial x} = -\frac{1}{c}a'_x = -\frac{1}{c}\frac{\partial a_x}{\partial t}$$

and

$$-\frac{1}{c}\frac{\partial \phi}{\partial t} = -\frac{1}{c}\phi' = \frac{\partial \phi}{\partial x}$$

where the superscript is used to indicate the derivative taken with respect to the argument $t - \frac{x}{c}$. Thus equation (7.2) may be written in the two different forms

$$\frac{\partial a_x}{\partial x} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial a_x}{\partial t} = \frac{\partial \phi}{\partial t}. \tag{7.3}$$

We have for solutions of these two partial differential equations

$$a_x = \phi + c_1(t)$$
$$a_x = \phi + c_2(x).$$

In comparing these two solutions we see that c_1 and c_2, functions of x and t, respectively, are mere constants and equal. We thus have

$$a_x = \phi + c_1$$

and the constant c_1 may be taken equal to zero, since it drops out in the determination of the electromagnetic intensities.

In rectangular coordinates the electromagnetic intensities, as defined by equations (5.1) and (5.2), take the form

$$e = -\frac{1}{c}(a_x i + a_y j + a_z k) - \frac{\partial \phi}{\partial x}i$$

$$h = \text{curl } a = -\frac{\partial a_z}{\partial x}j + \frac{\partial a_y}{\partial x}k$$

Also since

$$-\frac{1}{c}\dot{a}_x = -\frac{1}{c}a'_x = \frac{\partial a_x}{\partial x}$$

and

$$\frac{\partial a_x}{\partial x} = \frac{\partial \phi}{\partial x}$$

according to the first of equations (7.3), the x coordinate of the vector $\boldsymbol{e}$ vanishes; we thus have

$$\left.\begin{aligned} \boldsymbol{e} &= -\frac{1}{c}(\dot{a}_y\boldsymbol{j} + \dot{a}_z\boldsymbol{k}) \\ \boldsymbol{h} &= -\frac{\partial a_z}{\partial x}\boldsymbol{j} + \frac{\partial a_y}{\partial x}\boldsymbol{k}. \end{aligned}\right\} \tag{7.4}$$

If we now take the axis OY, parallel to the vector, $\boldsymbol{a}$, its coordinate, a_z, vanishes; and if we put $a_y = f\left(t - \frac{x}{c}\right)$, equations (7.4) become

$$\boldsymbol{e} = -\frac{1}{c}f'\left(t - \frac{x}{c}\right)\boldsymbol{j}$$

$$\boldsymbol{h} = -\frac{1}{c}f'\left(t - \frac{x}{c}\right)\boldsymbol{k}$$

or in rectangular coordinates

$$\left.\begin{aligned} e_x &= e_z = h_x = h_y = 0 \\ e_y &= -\frac{1}{c}f'\left(t - \frac{x}{c}\right) \\ h_z &= -\frac{1}{c}f'\left(t - \frac{x}{c}\right). \end{aligned}\right\} \tag{7.5}$$

The function a_x or ϕ does not enter this solution, so we may just as well assume that

$$a_x = \phi = 0.$$

We see from equations (7.5) that the electric and magnetic vectors are equal in magnitude, mutually perpendicular, and perpendicular to the velocity of propagation.

The electromagnetic intensities thus lie in the wave front Also, since there is no component of these intensities parallel to the velocity of propagation, we may conclude that electromagnetic disturbances are propagated as transverse waves. That electromagnetic waves, which include light waves as a special instance, are transverse waves, used to be regarded as an experimental fact. We must not conclude from this that the electric and magnetic vectors are always at right angles to each other and perpendicular to the velocity of propagation. In fact, this is not the situation at all in the general case

The equations defining a_y and therefore e_y and h_z are all wave equations The special forms of the function, f, are numerous; for waves of a periodic type we may assume as a special solution that a_y or f is given by the equation

$$a_y = -a \sin n\left(t - \frac{x}{c}\right)$$

Then equations (7 5) become

$$\left.\begin{aligned} e_y &= a \cos n\left(t - \frac{x}{c}\right) \\ h_z &= a \cos n\left(t - \frac{x}{c}\right) \end{aligned}\right\} \qquad (7.6)$$

where $\frac{n}{2\pi}$ is the frequency, and a, which is written here for $\frac{a_1 n}{c}$, is the amplitude of the vibrations This simple form represents a system of plane polarized light waves if the frequency is high enough. At the time $t = t_1$, a wave form will be in the plane $x = x_1$, and since c is the velocity of propagation, this same wave form will be in the plane $x = x_1 + cdt$, at the time $t = t_1 + dt$. At this instant there will be a new wave front, a different e_y, and a different h_z in the plane. In fact, in this plane the electromagnetic intensities go through periodic values ranging from $-a$ to a.

8. Kirchhoff's Delayed Potentials.—In electrostatics we were able to obtain the electrostatic potential as a solution of Poisson's equation in the form of a Newtonian potential function. We may obtain solutions of equations (5.5), and (5.6), in a similar form, by a method due to Kirchhoff. These solutions will be Newtonian potential functions at an instant, t, for the distribution and motion of the charges at the previous instant, $t - \frac{r}{c}$. This is a natural expectation, for the effects of a charge, distant r from a point P, would be appreciated at that point after the lapse of the time interval $\frac{r}{c}$, since electromagnetic disturbances are propagated in free space with the velocity, c, of light.

More specifically, then, we wish to show that the solution of the equations

$$\left.\begin{aligned} \Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho \\ \Delta' a - \frac{1}{c^2}\frac{\partial^2 a}{\partial t^2} &= -4\pi\rho u \end{aligned}\right\} \tag{8.1}$$

are given by the equations

$$\phi = \int\frac{[\rho]d\tau}{r}, \quad a = \int\frac{[\rho u]d\tau}{r} \tag{8.2}$$

where the square brackets are here used to denote the electric volume density and the convection current density at the instant $t = -\frac{r}{c}$. These potentials then characterize the electromagnetic field at a point, P, at the instant $t = 0$.

To obtain the solution of the first of the equations (8.1), we apply Green's theorem in its second form to the function ϕ, and a function ψ, satisfying the equation

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{8.3}$$

The point P, at which we wish to determine the potential ϕ, we will select within a closed region bounded by an arbitrary surface σ. If we now inclose the point P by a sphere, S, of infinitesimal radius a, the second form (19) of Green's theorem for the region τ, lying between the surface σ and the sphere S, becomes

$$\left.\begin{aligned}\int_\tau (\phi\Delta\psi - \psi\Delta\phi)d\tau &= \int_\sigma (\phi \operatorname{grad} \psi - \psi \operatorname{grad} \phi)\cdot n d\sigma \\ &+ \int_S (\phi \operatorname{grad} \psi - \psi \operatorname{grad} \phi)\cdot n d\sigma\end{aligned}\right\} \quad (8.4)$$

We must remember here that n is the outer unit normal at every point of the surfaces bounding the region τ; for the sphere S, then, n is directed towards the center.

If we substitute the values of $\Delta\phi$ and $\Delta\psi$ obtainable from equations (8.1) and (8.3), into the left member of (8.4), this member becomes

$$\frac{1}{c^2}\int_\tau\left(\phi\frac{\partial^2\psi}{\partial t^2} - \psi\frac{\partial^2\phi}{\partial t^2}\right)d\tau + 4\pi\int_\tau \rho\psi d\tau$$

the first integral of which expression may evidently be written in the form

$$\frac{1}{c^2}\frac{\partial}{\partial t}\int_\tau\left(\phi\frac{\partial\psi}{\partial t} - \psi\frac{\partial\phi}{\partial t}\right)d\tau.$$

If after this substitution we multiply equation (8.4) through by dt, and integrate between the arbitrary limits t_1 and t_2, we will have the equation

$$\left.\begin{aligned}\frac{1}{c^2}\int_\tau\left(\phi\frac{\partial\psi}{\partial t} - \psi\frac{\partial\phi}{\partial t}\right)d\tau\Bigg|_{t_1}^{t_2} + 4\pi\int_{t_1}^{t_2}dt\int_\tau \rho\psi d\tau \\ = \int_{t_1}^{t_2}dt\int_\sigma(\phi \operatorname{grad} \psi - \psi \operatorname{grad} \phi)\cdot n d\sigma \\ + \int_{t_1}^{t_2}dt\int_S(\phi \operatorname{grad} \psi - \psi \operatorname{grad} \phi)\cdot n d\sigma.\end{aligned}\right\} \quad (8.5)$$

When the solution ψ of equation (5.5) was a function of r and t alone, we saw (6.2) that this solution assumed a special form; in accordance with this we will assume that

$$\psi = \frac{1}{r} F\left(t + \frac{r}{c}\right)$$

where F, as formerly, is an arbitrary function. By a proper selection of the arbitrary function F, we will be able to obtain our solution from equation (8.5)

We now impose the conditions that the function $F(\epsilon)$, vanishes outside the time interval $0 \leq \epsilon \leq \delta$, where δ is a small finite quantity, and that inside this interval, $F(\epsilon)$ is so large that

$$\int_0^\delta F(\epsilon)d\epsilon = 1.^* \qquad (8.6)$$

Under these assumptions we are able to prove the following useful lemma:

Lemma.

If, for a fixed r, $t_1 + \frac{r}{c} < 0$ *and* $t_2 - \frac{r}{c} > \delta$ *then*

$$\underset{\delta \to 0}{\text{Limit}} \int_{t_1}^{t_2} XF\left(t + \frac{r}{c}\right)dt = X_{t = -\frac{r}{c}} = [X]$$

The proof of this lemma depends on the fact that the integral $\int_{t_1}^{t_2} F\left(t + \frac{r}{c}\right)dt$ is unity, and this is a consequence of the conditions imposed on the function $F(\epsilon)$. For if we replace $t + \frac{r}{c}$ by ϵ, we will have

$$\int_{t_1}^{t_2} F\left(t + \frac{r}{c}\right)dt = \int_{t_1+\frac{r}{c}}^{t_2+\frac{r}{c}} F(\epsilon)d\epsilon = \int_0^\delta F(\epsilon)d\epsilon = 1$$

---

* It is easy to construct such a function In fact these conditions are satisfied by the function $\frac{\pi f(\epsilon)}{2\delta} \cos \frac{\pi\epsilon}{2\delta}$, if $f(\epsilon)$ is defined as unity for $0 \leq \epsilon \leq \delta$ and zero outside this interval.

the integral from $t_1 + \frac{r}{c}$ to 0 and from δ to $t_2 + \frac{r}{c}$ are zero since $F(\epsilon)$ vanishes outside the interval $0 < \epsilon < \delta$. To evaluate the limit of the integral in the lemma we consider the extreme values X' and X'' of X in the interval, $0 \leq \epsilon \leq \delta$. The integral itself will lie between these extreme values, since the integral $\int_{t_1}^{t_2} F\left(t + \frac{r}{c}\right)dt$ is unity. Also since ϵ ranges over the interval $0 \leq \epsilon \leq \delta$, ϵ or $t + \frac{r}{c}$ becomes zero with δ, so that the extreme values $X'(\epsilon)$ and $X''(\epsilon)$ of $X(\epsilon)$ approach one and the same value, $X(0)$, with vanishing δ. We may thus write

$$\underset{\delta \to 0}{\text{Limit}} \int_{t_1}^{t_2} XF\left(t + \frac{r}{c}\right)dt = X_{\epsilon=0} = X_{t=-\frac{r}{c}} = [X]$$

We will now apply this lemma to the second integral in equation (8.5). When we replace ψ by $\frac{1}{r} F\left(t + \frac{r}{c}\right)$ this integral may be written in the form

$$4\pi \int_{t_1}^{t_2} dt \int_\tau \frac{\rho}{r} F\left(t + \frac{r}{c}\right) d\tau = \int_\tau d\tau \int_{t_1}^{t_2} \frac{4\pi\rho}{r} F\left(t + \frac{r}{c}\right) dt$$

The function X in this case is $\frac{4\pi\rho}{r}$, and according to the lemma the limit of the time integral as δ approaches zero is $\frac{4\pi[\rho]}{r}$, and the volume integral reduces to $4\pi \int \frac{[\rho] d\tau}{r}$. This will be the left member of equation (8.5), for the first integral vanishes at $t = t_1$ and $t = t_2$, since ψ and therefore also $\frac{\partial \psi}{\partial t}$ are identically zero outside the interval $0 < \epsilon < \delta$.

The second part of the integral over the surface σ,

$$-\int_{t_1}^{t_2} dt \int_\sigma \psi \text{ grad } \phi \cdot n d\sigma = -\int_\sigma d\sigma \int_{t_1}^{t_2} \frac{\text{grad } \phi \cdot n}{r} F\left(t + \frac{r}{c}\right) dt$$

$$= \int_\sigma \frac{[\text{grad } \phi] \cdot n d\sigma}{r}$$

since $\frac{1}{r}$ grad $\phi \cdot n$ plays the rôle of the function in the application of the lemma. Also since $\psi = \frac{1}{r} F\left(t + \frac{r}{c}\right)$

$$\text{grad } \psi = \frac{\partial \psi}{\partial r} \text{grad } r = -\frac{1}{r^2} F\left(t + \frac{r}{c}\right) \text{grad } r$$
$$+ \frac{1}{cr} F'\left(t + \frac{r}{c}\right) \text{grad } r$$

and the first part of the integral over the surface σ

$$\int_{t_1}^{t_2} dt \int_\sigma \phi \text{ grad } \psi \cdot n d\sigma = -\int_\sigma d\sigma \int_{t_1}^{t_2} \frac{\phi}{r^2} \text{grad } r \cdot n F\left(t + \frac{r}{c}\right) dt$$
$$+ \int_\sigma d\sigma \int_{t_1}^{t_2} \frac{\phi}{cr} \text{grad } r \cdot n F'\left(t + \frac{r}{c}\right) dt.$$

If we now integrate by parts the time integral involving $F'\left(t + \frac{r}{c}\right)$ as a factor, we will have

$$\int_{t_1}^{t_2} \frac{\phi \text{ grad } r \cdot n}{cr} F'\left(t + \frac{r}{c}\right) dt = F\left(t + \frac{r}{c}\right) \frac{\phi \text{ grad } r \cdot n}{cr} \Bigg|_{t_1}^{t_2}$$
$$- \int_{t_1}^{t_2} \frac{\partial \phi}{\partial t} \frac{\text{grad } r \cdot n}{cr} F\left(t + \frac{r}{c}\right) dt.$$

The first of these two integrals vanishes, since F is zero at the limits t_1 and t_2, while the second of these integrals becomes

$$-\left[\frac{\partial \phi}{\partial t}\right] \frac{\text{grad } r \cdot n}{cr},$$

upon the application of the lemma. Also according to the lemma the integral $-\int_\sigma d\sigma \int_{t_1}^{t_2} \frac{\phi \text{ grad } r \cdot n}{r^2} F dt$ becomes

$$-\int_\sigma \frac{[\phi] \text{ grad } r \cdot n}{r^2} d\sigma$$

so that the complete integral over the surface σ takes the form

$$-\int_\sigma \frac{[\phi]\,\mathrm{grad}\,r\cdot n}{r^2}\,d\sigma - \int_\sigma \left[\frac{\partial\phi}{\partial t}\right]\frac{\mathrm{grad}\,r\cdot n}{cr}\,d\sigma$$
$$-\int_\sigma \frac{\mathrm{grad}\,\phi\cdot n}{r}\,d\sigma \qquad (8.7)$$

The integral over the surface of the sphere S will assume the same form as the integral over the surface σ. The two integrals involving $\frac{1}{r}$ as a factor and taken over the sphere S will evidently approach zero when the radius of the sphere approaches zero. If we write $[\phi']$ and $[\phi'']$ for the extreme values of $[\phi]$ on the surface of the sphere, then the remaining integral

$$-\int_S \frac{[\phi]\,\mathrm{grad}\,r\cdot n}{r^2}\,d\sigma = \int_S \frac{[\phi]d\sigma}{r^2}\,*$$

over the surface of the sphere will lie between $4\pi[\phi']$ and $4\pi[\phi'']$. But $[\phi']$ and $[\phi'']$ approach one and the same value, $[\phi]$, at the point P, as the radius a approaches zero. Also in this case $t = -\frac{r}{c} = -\frac{a}{c}$ goes to zero with the radius of the sphere, so that the square brackets have lost their significance and may be omitted. We thus see that the integral over the surface of the sphere is equal to $4\pi\phi(P)$.

If we now choose our surface, σ, as a very large sphere, radius R, with center at the point P, then, by a very mild assumption about the function ϕ, we will be able to show that as the radius of the sphere is made to increase without limit the surface integrals (8.7) vanish. We will assume that up to a certain time, t, the function ϕ was identically zero in the infinitely remote regions of space. Then since in these integrals ϕ is to be determined at the time $t = -\frac{R}{c}$,

* At the surface of the sphere grad r and n are unit vectors of opposite sense, so that their scalar product is -1.

which for a sufficiently large R is previous to the instant t_1, the integrands of all of these integrals are identically zero; thus the integrals are always zero, and we have, finally, as R increases without limit,

$$\phi = \int_\tau \frac{[\rho]d\tau}{r}$$

It will be evident from this that the vector potential a will have a similar form * or the electromagnetic potentials have the forms given in equations (8 2).†

9. Ether Stresses and the Electromagnetic Momentum.—Another idea similar to the idea expressed by the flow of energy is Abraham's *electromagnetic momentum* This is arrived at by considering the resultant electromagnetic force on a group of electric corpuscles, possibly constituting a ponderable body in a particular electromagnetic state The resultant force on a region τ, bounded by a surface σ, containing the corpuscles on its interior, will be given by the integral

$$F = \int_\tau \rho\left(e + \frac{1}{c} u \times h\right)d\tau \tag{9.1}$$

By an application of the field equations (2 1), this integral may be put into a more significant form We will replace the electric volume density ρ, by $\frac{1}{4\pi}$ div e, and the conduction current density ρu, by its equal,

$$\frac{1}{4\pi}(c \operatorname{curl} h - \dot{e}),$$

* From the equation

$$\Delta' a - \frac{1}{c^2}\frac{\partial^2 a}{\partial t^2} = -4\pi\rho u$$

we see that

$$\Delta a_x - \frac{1}{c^2}\frac{\partial^2 a_x}{\partial t^2} = -4\pi\rho u_x$$

or

$$a_x = \int \frac{[\rho u_x]d\tau}{r}$$

† A variety of forms for the electromagnetic potentials are given by Schott in his book on "Electromagnetic Radiation "

so that our integral may be written in the form

$$F = \frac{1}{4\pi}\int_\tau \left(\operatorname{div} e\cdot e - \frac{1}{c}\, e \times h - h \times \operatorname{curl} h\right) d\tau$$

We note first that

$$\frac{\partial}{\partial t}(e \times h) = e \times h + e \times h$$

or

$$\dot{e} \times h = \frac{\partial e \times h}{\partial t} + ce \times \operatorname{curl} e$$

when we replace h by its value from the field equations (2.1). Also since div h is identically zero we may introduce the additive term (div h)h, into the integrand for symmetry. If we introduce this term and substitute the value for $e \times h$ just obtained, we can break F up into two components so that

$$F = F_1 + F_2$$

where

$$\left.\begin{aligned} F_1 &= \frac{1}{4\pi}\int_\tau [(\operatorname{div} e)e - e \times \operatorname{curl} e \\ &\qquad + (\operatorname{div} h)h - h \times \operatorname{curl} h)]d\tau \\ F_2 &= -\frac{1}{c}\frac{\partial}{\partial t}\int_\tau \frac{e \times h d\tau}{4\pi} = -\frac{1}{c^2}\frac{\partial}{\partial t}\int_\tau s d\tau \end{aligned}\right\} \quad (9\ 2)$$

and

To analyze the characteristic properties of the ponderomotive force F, acting on the electromagnetic system inside the surface σ, we may first consider the system to be in a steady state. The force component F_2 vanishes, since s is constant with time; the resultant force is then the single component F_1, which may be transformed into the surface integral

$$F_1 = \frac{1}{8\pi}\int_\sigma [2e\ ne - e^2 n + 2h\cdot nh - h^2 n]d\sigma \ *$$

* This transformation theorem is beyond the elements of vector analysis, its validity may be demonstrated by transforming one component of the electric vector into a surface integral For the x component we have

Thus the quantity in square brackets divided by 8π is the stress across the surface element $d\sigma$, due to the ether outside the bounding surface. For brevity we will consider the stress components due to the electric intensity. We will adopt the usual notation X_n, Y_n, Z_n for the stress components parallel to the indicated axes across the element whose normal is indicated by the index. If we project the resultant stress across the element on OX we will get

$$8\pi X_n = 2e\ ne_x - e^2 \cos(nx)$$
$$= 2e_n e_x - e^2 \cos(nx)$$

If the element is taken normal to OX then X_n changes to X_x or

$$8\pi X_x = 2e_x{}^2 - e^2$$

Also since

$$8\pi Y_n = 2e\ ne_y - e^2 \cos(ny)$$

$$\int_\tau \left[\left(\frac{\partial e_x}{\partial x} + \frac{\partial e_y}{\partial y} + \frac{\partial e_z}{\partial z}\right)e_x + \left(\frac{\partial e_x}{\partial z} - \frac{\partial e_z}{\partial x}\right)e_z - \left(\frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y}\right)e_y\right]d\tau$$

$$= \tfrac{1}{2}\int_\tau \left[\frac{\partial}{\partial x}(e^2_x - e^2_y - e^2_z) + \frac{\partial}{\partial y}(2e_x e_y) + \frac{\partial}{\partial z}(2e_x e_z)\right]d\tau$$

and by Gauss's theorem

$$= \tfrac{1}{2}\int_\sigma [(e^2_x - e^2_y - e^2_z)\cos(xn) + 2e_x e_y \cos(yn) + 2e_x e_z \cos(zn)]d\sigma$$

$$= \tfrac{1}{2}\int_\sigma [2e_x\{e_x \cos(xn) + e_y \cos(yn) + e_z \cos(zn)\} - e^2 \cos(xn)]d\sigma$$

$$= \tfrac{1}{2}\int_\sigma [2e_x e_n - e^2 \cos(xn)]d\sigma$$

Thus the vector form for the integrand is composed of the two component vectors

$$2e_n(e_x i + e_y j + e_z k) = 2e\ ne$$

and

$$-e^2[\cos(xn)i + \cos(yn)j + \cos(zn)k] = -e^2 n$$

The general transformation theorem is therefore

$$\int_\tau (\operatorname{div} uu - u \times \operatorname{curl} u)d\tau = \tfrac{1}{2}\int_\sigma (2u\ nu - u^2 n)d\sigma.$$

it follows that

$$8\pi Y_x = 2e_x e_y - e^2 \cos(xy) = 2e_x e_y$$

and

$$8\pi Z_x = 2e_x e_z$$

Thus $X_x Y_x Z_x$ appear as the components of the vector T_x, the stress across the element normal to OX In just the same way we may find the stress components across elements in the zx and xy planes, or we may permute the indices. There will be just six of these components, since $Y_x = X_y$, etc These components define the stress at a point in the sense that the stress across every plane through the point may be determined in terms of the stresses across three mutually perpendicular planes. This theorem is not difficult to believe, since we have used its converse in deducing the six components of stress

If we introduce the energy density

$$w \equiv \frac{e^2 + h^2}{8\pi}$$

and include the magnetic terms, the stresses across the yz plane will have the form

$$X_x = \frac{e_x^2 + h_x^2}{4\pi} - w; \quad Y_x = \frac{e_x e_y + h_x h_y}{4\pi}, \quad Z_x = \frac{e_x e_z + h_x h_z}{4\pi}$$

Or the complete stress system may be written in matrix form

$$\begin{Vmatrix} X_x X_y X_z \\ Y_x Y_y Y_z \\ Z_x Z_y Z_z \end{Vmatrix} \equiv \begin{Vmatrix} e_x^2 + h_x^2 - w, & e_x e_y + h_x h_y, & e_x e_z + h_x h_z \\ e_x e_y + h_x h_y, & e_y^2 + h_y^2 - w, & e_y e_z + h_y h_z \\ e_x e_z + h_x h_z, & e_y e_z + h_y h_z, & e_z^2 + h_z^2 - w \end{Vmatrix} \quad (9.3)$$

where we have finally used (i.e.s.) units to drop out the 4π. This is the stress tensor, or in our particular situation, the electromagnetic stress tensor. This tensor, first deduced by Maxwell, seems to play an important rôle in general relativity theory.

We are not insisting on the reality of the ether stresses; but we can use the language of mechanics more freely Thus, in the case considered, the ponderomotive force F may be thought of as due to a system of ether stresses over the bounding surface, and in the general case we see that these stresses form only a part of the force acting on the electromagnetic system

Suppose next that the surface σ contains no electric charges in its interior, then the density $\rho = 0$, and as a consequence $F = 0$, or

$$F_1 = - F_2 = \frac{1}{c^2}\frac{\partial}{\partial t}\int s d\tau.$$

We thus find that there is a non-zero force acting on every volume element of the ether Or no volume element of the ether is in equilibrium under the stress system acting on its boundary.

If now we restrict our considerations to a system of finite dimensions, which is always the actual situation in experiment, then the bounding surface σ may be any arbitrary surface inside of which lies the system of corpuscles. This is true since the integrand of the integral (9 1) vanishes identically at points of the region τ, where the density ρ is zero We will now transform the volume integral (9.2) into a surface integral, and let the arbitrary bounding surface σ recede to infinity in all directions; it will be seen that this surface integral or the component F_1 vanishes.

Since e is a Newtonian force intensity and vanishes at infinity like $\frac{1}{r^2}$, the integrand of the surface integral, involving the electric intensity, vanishes at infinity like $\frac{1}{r^4}$ We may thus conclude, and the detailed proof is similar to that already used (II, 8), that this surface integral vanishes if we include all space as our region, τ. This same argument is applicable to the surface integral con-

taining the magnetic intensity h in the integrand The force component F_1 is thus zero, and the force F reduces to

$$F = F_2 = -\frac{1}{c^2}\frac{\partial}{\partial t}\int_\infty s\,d\tau$$

where the integration is to include all space.

If we write

$$G = \frac{1}{c^2}\int_\infty s\,d\tau \tag{9 4}$$

where the subscript is to indicate as before that the region of integration is entire space, then since

$$F = -\frac{\partial G}{\partial t} \tag{9 5}$$

the vector G has the characteristic property of momentum, namely, its time rate of change is force. It is this quantity, G, that is referred to as the electromagnetic momentum The force F is thus closely related to the flow of energy; in fact, the flow of energy in every volume element contributes its part to the integral

10. Radiation Pressure.—We shall apply the idea of electromagnetic momentum to the problem of radiation pressure To do this we consider a source of light sending out rays in a single direction; the source will thus be far removed, so that the light rays will be parallel rays.

Suppose this radiation enters a cylinder of right section σ, whose elements are parallel to the light rays. Then the energy that crosses a section of the cylinder is equal to the flux of the vector radiant s, through this section. To compute this we will assume that s at every point of the section may be replaced by its average value $\bar{s}$, taken for a whole period; then since the radiation is traveling with the speed c, the radiation crossing the section σ, in the time dt, will evidently be

$$|\bar{s}|c\sigma dt$$

cdt being the altitude of the cylinder containing the radiation under consideration. But $|G|$ is this energy multiplied by $\frac{1}{c^2}$ (9.4) and

$$|G| = \frac{1}{c^2}|\bar{s}|c\sigma = |F|$$

so we see that $|\bar{s}|c\sigma$, the radiant energy per unit of time, is just equal to $c^2|F|$ or

$$|F| = \frac{1}{c}|\bar{s}|\sigma$$

Thus if the energy is intercepted by a black disk the pressure on the disk will be given by the above equation

For plane polarized light we may easily compute the average value $|\bar{s}|$. We will take the axis OX as the direction of propagation, and $\boldsymbol{e}$ and $\boldsymbol{h}$ parallel to the OY and OZ axes, respectively, then (7.6)

$$e_y = a \cos n\left(t - \frac{x}{c}\right)$$

$$h_z = a \cos n\left(t - \frac{x}{c}\right)$$

and

$$\frac{c}{4\pi}|\boldsymbol{e} \times \boldsymbol{h}| = \frac{c|e_y h_z|}{4\pi} = |s|$$

Hence

$$|\bar{s}| = \frac{n}{2\pi}\int_0^{\frac{2\pi}{n}}|s|dt = \frac{nca^2}{8\pi^2}\int_0^{\frac{2\pi}{n}}\cos^2 n\left(t - \frac{x}{c}\right)dt = \frac{ca^2}{8\pi}$$

or the pressure on the disk is $\frac{a^2}{8\pi}$

SPECIAL RELATIVITY AND THE ELECTRODYNAMIC THEORY

11. The Michelson–Morley Experiment.—At the time of the Michelson–Morley experiment physicists in general entertained the concept of a stagnant ether which percolated undisturbed through a moving body. These are added conditions on the ether already defined as a medium supporting the propagation of electromagnetic disturbances. It was expected that the stagnant ether would furnish a fixed frame of reference with respect to which the absolute velocity of the earth through space could be determined.

Just how the physicist arrived at this conception of the ether is a long story not pertinent here But if we hypothecate such an ether we will be able to discuss the theory of the Michelson–Morley experiment and the consequences which followed its failure.

The apparatus used in this experiment consisted, in part, of two equal arms, OA and OB, at right angles. Instead of considering the apparatus as moving through the ether we may regard the ether as drifting past the apparatus with the reverse velocity u of the earth Then if OA is turned parallel and opposite to the ether stream it will be easy to show that it will take longer for a ray of light to travel from O to A and back than to go from O to B and back across the ether current.

If $OA = OB = a$, then the time for the trip from O to A is $\frac{a}{c - u}$, where c is the velocity of light and u the velocity of the ether relative to the apparatus; the time of the return trip will evidently be $\frac{a}{c + u}$, since the light and the ether are traveling in the same direction, or the time of the whole trip is

$$t_1 = \frac{a}{c - u} + \frac{a}{c + u} = \frac{2ac}{c^2 - u^2}.$$

In computing the time t_2 of the trip from O to B and return, the light ray must have a component velocity u, parallel to the motion of the ether, so that the velocity along the arm is $\sqrt{c^2 - u^2}$, then

$$t_2 = \frac{2a}{\sqrt{c^2 - u^2}}.$$

If we write

$$t_1 = \frac{2a\beta^2}{c} \tag{11.1}$$

and

$$t_2 = \frac{2a\beta}{c} \tag{11.2}$$

where

$$\beta = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

a quantity greater than unity, we see at once that $t_1 > t_2$.

When the experiment was tried the interferometer exhibited no interference bands Or the experiment showed conclusively that t_1 and t_2 were equal. Thus the ether failed entirely to function as a fixed frame of reference. It would seem that the natural thing would have been to abandon the ether hypothesis altogether, but at this time the electrical theory of matter was being evolved, and the work of Lorentz and Larmor on the electron theory, which even hypothecates the electron as a modified form of the ether, seemed very promising Then physicists were addicted to the undulatory theory of light, and they simply had to have a medium for the transmission of electromagnetic waves At any rate, since the ether hypothesis was retained, it was left to the theorist to obviate the inconsistency in the theory and the result of the Michelson–Morley experiment.

One way out of this dilemma was the suggestion that the ether might be convected along with the earth; another was that the velocity of the source relative to the observer modified the velocity of light for the observer But the

suggestion of Fitzgerald, that the ether percolating through the arm parallel to the ether drift must have caused a contraction of the arm, was the one that survived and was utilized by Lorentz.

Thus if $OB = a$ and we take $OA = \frac{a}{\beta}$, according to the Fitzgerald hypothesis, then,

$$t_2 = \frac{2a\beta}{c} \quad \text{and} \quad t_1 = \frac{2a}{c\beta}\beta^2$$

or $t_1 = t_2$, which is in accord with experimental results. Or, what is the same thing, if either arm is of length a when perpendicular to the ether drift, it contracts to $\frac{a}{\beta}$ when turned parallel to it

12. The Lorentz Transformation Equations.—Physicists had believed for a long time that the velocity of light is the same at every point of free space and in every direction. That the velocity of light might be the same for an observer fixed in the ether and one moving uniformly through it, as the failure of the Michelson–Morley experiment seemed to indicate, suggests that the Maxwell equations might have the same form for two such observers. This was the point of view of Lorentz.

Under the hypotheses of a stagnant ether and a Fitzgerald contraction we proceed to find out how the coordinates of a point (x, y, z) referred to a frame of reference fixed in the ether, are related to the coordinates (x', y', z') of the same point referred to a frame of reference moving with a uniform velocity u through it. We shall frequently refer to the space defined by the coordinates (x, y, z) as the space S, and the superimposed space defined by the coordinates (x', y', z') as the space S'.

For simplicity we shall choose the $O'X'$ and the OX axes coincident, and suppose that the space S' is moving in the direction OX with the velocity u, so that $y' = y$ and $z' = z$. Then if at the time $t = 0$, S observes that O' is at O, he will find that the distance from the point

(x, y, z) to the $y'z'$ plane will be $x - ut$. He also knows that the measuring stick used by S' in measuring this distance is shortened in the ratio of $\frac{1}{\beta}$ according to the contraction hypothesis, so that x', the measurement of S', will be β times too large. Thus

$$x' = \beta(x - ut). \tag{12.1}$$

To determine how S and S' measure time, we first time a light signal over the double journey OA. The observer S' moving through the ether with the velocity u of the earth, and not aware of his motion, estimates the trip as $2a$ and the time $\frac{2a}{c}$, but S, who knows that the arm OA is in uniform motion, uses the contracted length $\frac{2a}{\beta}$ and figures the double journey, according to (11.1), to be $t = \frac{2a}{\beta c}\beta^2$. We thus find that

$$t = \beta t' \tag{12.2}$$

or the units of time used by S' are larger than those used by S

There is another difference in their time measurements. The observer S measures the trip from O to A as $\frac{a}{\beta(c - u)}$, in his own units If he uses the units of S' in measuring the same trip he has to divide this by β, according to (12.2), or

$$\frac{a}{\beta^2(c - u)} = \frac{a}{c} + \frac{au}{c^2}.$$

But S' figures the time of this trip to be $\frac{a}{c}$; thus if S', located at A, wishes to set his clock with the clock of S at O, he will add $\frac{a}{c}$ to the reading he gets to allow for the time it takes the signal to reach him But S, figuring the time

of the trip as $\frac{a}{c} + \frac{au}{c^2}$ in S''s units, knows that S''s clock has been set $\frac{au}{c^2}$ units too slow measured in S''s own units Thus if S' measures the distance x' at the time t in his own units S will correct the time, making it $t' + \frac{x'u}{c^2}$ When we introduce the correction term $\frac{x'u}{c^2}$ into (12 2) we will have

$$t = \beta\left(t' + \frac{x'u}{c^2}\right). \tag{12 3}$$

If we substitute (12.3) into (12 1), we will find that

$$x = \beta(x' + ut')$$

so that

$$x = \beta(x' + ut'), \quad y = y', \quad z = z', \quad t = \beta\left(t' + \frac{ux'}{c^2}\right) \tag{12.4}$$

furnishes the space time relations between the two observers. Also, since S's measurements will not change if he is moving through the ether, (12.4) will still hold providing S' is moving relative to S with the uniform velocity u. This set of equations is due to Lorentz, who introduced t' as some sort of fictitious time.

For S to say that his clock registers a certain time which is simultaneous with the reflection of the light at A has no meaning, unless S knows the velocity of the place A The observer at O figures the time of the event at A in terms of $\frac{au}{c^2}$ The old idea of simultaneity involved a time and two places, now it requires an additional knowledge of a relative velocity

13. The Relativity Viewpoint.—The general relativity point of view of Einstein may best be summarized by the following postulate:

It is possible to so formulate a physical law that the formulation is independent of the choice of a coordinate

system. Or more mathematically the statement of a physical law is a covariant relation

The first fundamental postulate of special relativity is a special instance of the postulate of general relativity; it may be briefly stated as follows

Physical laws or the equations defining them expressed in one rectangular cartesian coordinate system should take the same form when referred to a second rectangular cartesian coordinate system moving uniformly relative to the first without rotation.

This is really the conclusion of Lorentz and Larmor, while it is just the starting-point for Einstein

The second fundamental postulate of "special" relativity is possibly a natural conclusion to be drawn from the failure of the Michelson–Morley experiment. It may be put into the concise form·

The velocity of light is the same for all observers.

This hypothesis that the velocity of light is constant is a universal law, though possibly only approximately true In fact we may state, parenthetically, that in general relativity this postulate is thrown out entirely

This second postulate throws suspicion on the hypotheses used in the deduction of the Lorentz transformation equations (12 4), wherein the velocity of the observer was added to the velocity of light However, the equations are correct under the special relativity hypotheses, and may be deduced directly from the second postulate We will formulate the problem from the new point of view.

If a point of light is moving with velocity c, and after the time interval dt its space coordinates have changed by dx, dy, dz, then S figures that

$$dx^2 + dy^2 + dz^2 = c^2dt^2 \tag{13.1}$$

while if dx', dy', dz', dt' are the corresponding space time intervals for the frame S', then

$$dx'^2 + dy'^2 + dz'^2 = c^2dt'^2 \tag{13 2}$$

since the velocity of light is the same for both observers

The problem now arising is a purely mathematical one How must the space time coordinates of the two frames be related so that (13 1) and (13.2) will hold? It is not difficult to analyze this problem, the essential non-trivial solutions are contained in the Lorentz transformation equations. Since we already have the solution we will utilize it, then if the Lorentz transformations (12 4) are substituted into (13.1), equation (13.2) will be the immediate consequence; this furnishes an adequate verification of the solution of the problem.

One of the first observations of Einstein was that the Lorentz equations (12 4), when solved for the coordinates x', y', z', t', express the symmetrical relation

$$x' = \beta(x - ut), \quad y' = y, \quad z' = z; \quad t' = \beta\left(t - \frac{ux}{c^2}\right) \qquad (13\ 3)$$

which just reverses the situation of the two observers. The t' appears now as S''s local time. Accordingly S' may assert that S is in motion with velocity $-u$, that his measuring rods are shortened, and his clocks are slow. Each observer may make these assertions, and there is no way of knowing which is correct. We must now look on t and t' as each observer's local time. Also it is the relative velocity of the two observers which plays the important rôle, and not the percolation of a stagnant ether The deduction of the Lorentz transformation equations from the second postulate in no way involves the hypothesis of a stagnant or any other kind of ether

Thus the Fitzgerald contraction may no longer be looked upon as a property of matter due to the percolating ether, but rather a relation between the observer and matter. For any observer at rest relative to a body, its length has its maximum value; for an observer in uniform motion parallel to the length of the body, its length becomes its original length divided by β, which length approaches zero as the velocity of the body approaches the velocity of light.

From the new point of view, whether general relativity or special relativity, the question of the existence of the ether does not arise, for in relativity theory we strive to formulate physical laws in such a way that they will be valid for every coordinate system. Thus relativity theory furnishes us with a criterion for determining whether a physical law is correctly formulated or not We proceed to apply this to the equations of the electromagnetic field.

14 Lorentz's Equations for Free Space Correlated for the Spaces S and S'.—We wish now to show how Maxwell's equations for free space are invariant under the Lorentz transformation (12 4) Or we shall show that observers in the spaces S and S' will use the same form for the defining equations of electromagnetic phenomena.

In changing from the coordinates of the frame S, to those of frame S', two formulae which we proceed to deduce will be found convenient Since x and t are functions of x' and t' (12 4), we may write

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t}$$

But from equations (13.3) we see that

$$\frac{\partial x'}{\partial x} = \beta, \quad \frac{\partial t'}{\partial x} = -\frac{\beta v}{c^2}$$

and

$$\frac{\partial t'}{\partial t} = \beta, \quad \frac{\partial x'}{\partial t} = -\beta v$$

so we find that

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \beta\left(\frac{\partial f}{\partial x'} - \frac{v}{c^2}\frac{\partial f}{\partial t'}\right) \\ \frac{\partial f}{\partial t} &= \beta\left(\frac{\partial f}{\partial t'} - v\frac{\partial f}{\partial x'}\right) \end{aligned}\right\} \qquad (14\ 1)$$

which are the formulae we wished to deduce. It should be observed that the relative velocity v has replaced the u used before.

We now write the first of Maxwell's equations for free space in the coordinate form

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial e_x}{\partial t} &= \frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z} \\ \frac{1}{c}\frac{\partial e_y}{\partial t} &= \frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x} \\ \frac{1}{c}\frac{\partial e_z}{\partial t} &= \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} \end{aligned}\right\} \tag{14.2}$$

If we apply formulae (14 1) to the second and third of (14 2), and transpose terms, we will easily find that

$$\frac{1}{c}\frac{\partial}{\partial t'}\beta\left(e_y - \frac{v}{c}h_z\right) = \frac{\partial h_x}{\partial z'} - \frac{\partial}{\partial x'}\beta\left(h_z - \frac{v}{c}e_y\right)$$

and

$$\frac{1}{c}\frac{\partial}{\partial t'}\beta\left(e_z + \frac{v}{c}h_y\right) = -\frac{\partial h_x}{\partial y'} + \frac{\partial}{\partial x'}\beta\left(h_y + \frac{v}{c}e_z\right)$$

Suppose we designate the electric and magnetic intensities as measured in the frame S' by e' and h', respectively. We then see that, if the defining equations have the same form in S' that they have in S, then it follows that

$$e'_y = \beta\left(e_y - \frac{v}{c}h_z\right)$$

$$e'_z = \beta\left(e_z + \frac{v}{c}h_y\right)$$

and

$$h'_z = \beta\left(h_z - \frac{v}{c}e_y\right)$$

$$h'_y = \beta\left(h_y + \frac{v}{c}e_z\right).$$

We easily solve these equations for h_y and h_z, finding that

$$h_y = \beta\left(h'_y - \frac{v}{c} e'_z\right)$$

$$h_z = \beta\left(h'_z + \frac{v}{c} e'_y\right)$$

If we substitute these values for h_y and h_z back into the first of equations (14.2), and replace $\frac{\partial e_x}{\partial t}$ by its equal,

$$\beta\left(\frac{\partial e_x}{\partial t'} - v\frac{\partial e_x}{\partial x'}\right)$$

we obtain the first of (14 2) in the form.

$$\frac{1}{c}\left[\frac{\partial e_x}{\partial t'} - v\left(\frac{\partial e_x}{\partial x'} + \frac{\partial e'_y}{\partial y'} + \frac{\partial e'_z}{\partial z'}\right)\right] = \frac{\partial h'_z}{\partial y'} - \frac{\partial h'_y}{\partial z'}. \quad (14.3)$$

We at once conclude, for an invariant form of Lorentz's equations for free space, that in addition to the relations already found we must have

$$e_x = e'_x$$

and

$$\operatorname{div} \boldsymbol{e}' = 0$$

If we substitute these relations together with the relation

$$h_x = h'_x$$

into the second of Lorentz's equations (2.1) we will find that it is identically satisfied providing we impose the condition that div $\boldsymbol{h}' = 0$

Instead of doing this, which is a simple matter of direct substitution, we will make a table of our results and draw our conclusions We have first

$$\left.\begin{aligned} e_x = e'_x;\ e_y &= \beta\left(e'_y + \frac{v}{c}h'_z\right),\ e_z = \beta\left(e'_z - \frac{v}{c}h'_y\right) \\ h_x = h'_x,\ h_y &= \beta\left(h'_y - \frac{v}{c}e'_z\right);\ h_z = \beta\left(h'_z + \frac{v}{c}e'_y\right) \end{aligned}\right\} \quad (14.4)$$

and solving we get

$$\left.\begin{aligned} e'_x &= e_x;\ e'_y = \beta\left(e_y - \frac{v}{c}h_z\right);\ e'_z = \beta\left(e_z + \frac{v}{c}h_y\right) \\ h'_x &= h'_x;\ h'_y = \beta\left(h_y + \frac{v}{c}e_z\right);\ h'_z = \beta\left(h_z - \frac{v}{c}e_y\right) \end{aligned}\right\} \quad (14.5)$$

while the field equations

$$\frac{1}{c}\frac{\partial e'}{\partial t'} = \operatorname{rot} h'$$

$$-\frac{1}{c}\frac{\partial h'}{\partial t'} = \operatorname{rot} e'$$

$$\operatorname{div} e' = 0$$

$$\operatorname{div} h' = 0$$

for the frame S' are identical in form with those for the frame S.

Equations (14.4) and (14.5) tell us that when

$$v = 0, \text{ then } e = e', \quad \text{and} \quad h = h'.$$

An observer in S will assert that relations (14.4) are valid when the frame S' is moving with velocity v, while an observer in S' may assert that (14.5) is valid when S is moving with the velocity $-v$. Also since (14.4) implies (14.5), and conversely, and since the field equations for observers in S and S' are identical in form, either may assert that his equations of the electromagnetic field in free space are the true ones, and there is no way of knowing which is correct.

15. The Correlation When a Convection Current is Present.—If there is a convection current density, ρu, present, then the transformed equation (14.3) carries the additional term $\frac{4\pi}{c}\rho u_x$, in its left member; it will then read

$$\frac{\beta}{c}\frac{\partial e_x}{\partial t'} - \frac{\beta v}{c}\left(\frac{\partial e'_x}{\partial x'} + \frac{\partial e'_y}{\partial y'} + \frac{\partial e'_z}{\partial z'}\right) + \frac{4\pi\rho u_x}{c}$$
$$= \beta\left(\frac{\partial h'_z}{\partial y'} - \frac{\partial h'_y}{\partial z'}\right) \quad (15.1)$$

while the equation

$$\operatorname{div} \boldsymbol{e} = 4\pi\rho$$

becomes

$$\beta\left(\frac{\partial e'_x}{\partial x'} - \frac{v}{c^2}\frac{\partial e'_x}{\partial t'}\right) + \beta\left(\frac{\partial e'_y}{\partial y'} + \frac{v}{c}\frac{\partial h'_z}{\partial y'}\right) + \beta\left(\frac{\partial e'_z}{\partial z'} - \frac{v}{c}\frac{\partial h'_y}{\partial z'}\right) = 4\pi\rho \qquad (15.2)$$

If we multiply (15 2) by $\frac{v}{c}$ and add to (15 1), and multiply (15 1) by $\frac{v}{c}$ and add to (15 2), we obtain, respectively, the two equations

$$\left.\begin{aligned} \frac{1}{c}\left[\frac{\partial e'_x}{\partial t'} + 4\pi\beta\rho(u_x - v)\right] &= \frac{\partial h'_z}{\partial y'} - \frac{\partial h'_y}{\partial z'} \\ \operatorname{div} \boldsymbol{e}' &= 4\pi\rho\beta\left(1 - \frac{vu_x}{c^2}\right). \end{aligned}\right\} \qquad (15.3)$$

For an invariant form in this more general case we see from these last equations that the following two equations must hold

$$\left.\begin{aligned} \beta\rho(u_x - v) &= u'_x\rho' \\ \beta\rho\left(1 - \frac{vu_x}{c^2}\right) &= \rho'. \end{aligned}\right\} \qquad (15.4)$$

We proceed to show that these equations are valid. A direct deduction of the first one will convince us that the two equations are not independent When we differentiate the first and fourth of (13 3) we get

$$dx' = \beta(dx - v\,dt)$$

$$dt' = \beta\left(dt - \frac{v}{c^2}\,dx\right)$$

while the differential quotient

$$\frac{dx'}{dt'} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

or

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}.$$

If we solve this equation for $u_x - v$ and multiply by $\beta\rho$ we find that

$$\beta\rho(u_x - v) = \beta\rho\left(1 - \frac{vu_x}{c^2}\right)u'_x$$

which is the first of (15 4) provided the second of (15 4) is true, i.e, provided that

$$\rho' = \beta\rho\left(1 - \frac{vu_x}{c^2}\right)$$

We will prove in the next article that this equation is true, let us assume for the present that it is valid. Then the term $\frac{1}{c}4\pi\beta\rho(u_x - v)$ becomes $\frac{1}{c}4\pi\rho' u'_x$, while the second equation will read

$$\operatorname{div} \mathbf{e}' = 4\pi\rho'$$

In this case of the convection current the second and third of equations (14.2) carry the additive terms

$$\frac{1}{c}4\pi\rho u_y \quad \text{and} \quad \frac{1}{c}4\pi\rho u_z$$

respectively Now

$$u_y = \frac{dy}{dt} = \frac{dy'}{dt} = \frac{dy'}{dt'}\frac{dt'}{dt} = \beta\left(1 - \frac{u_x v}{c^2}\right)u'_y$$

since

$$dt' = \beta\left(dt - \frac{vdx}{c^2}\right)$$

and similarly

$$u_z = \beta\left(1 - \frac{u_x v}{c^2}\right)u'_z.$$

If we now substitute these values of u_y and u_z in the additive terms mentioned above and use the fact that ρ' is to be equal to $\beta\rho\left(1 - \frac{vu_x}{c^2}\right)$, they will be seen to reduce to $\frac{1}{c}\,4\pi\rho' u'_y$ and $\frac{1}{c}\,4\pi\rho' u'_z$, respectively.

The field equations for the space S' in this case thus become

$$\frac{1}{c}\left(\frac{\partial e'}{\partial t'} + 4\pi\rho' u'\right) = \text{curl } h'$$

$$-\frac{1}{c}\frac{\partial h'}{\partial t'} = \text{curl } e'$$

$$\text{div } e' = 4\pi\rho'$$

$$\text{div } h' = 0$$

and again there is no way of determining which observer has the true defining equations.

16. The Invariance of Charge Hypothesis.—We will show here that, under the hypothesis that the electric charge is invariant under the Lorentz transformation,

$$\rho' = \beta\rho\left(1 - \frac{vu_x}{c^2}\right).$$

This will complete the demonstration of the previous article wherein the field equations in the electron theory were correlated for the spaces S and S'.

An observer at rest relative to a volume element measures the element as $d\tau_0$; an observer in the frame S, who sees the element in motion with the translatory velocity u, measures the same element as $d\tau$, but since the second observer is really moving with velocity u relative to the element, he must correct his measurement parallel to the direction of motion or

$$d\tau_0 = d\tau\left(1 - \frac{u^2}{v^2}\right)^{-\frac{1}{2}}.$$

Similarly an observer seeing the element in motion with velocity u', measures the element as $d\tau'$, and correcting his measurements we find that

$$d\tau_0 = d\tau'\left(1 - \frac{u'^2}{c^2}\right)^{\frac{1}{2}}$$

Thus

$$\frac{d\tau'}{d\tau} = \frac{(1 - u^2)^{\frac{1}{2}}}{(1 - u'^2)^{\frac{1}{2}}}$$

But

$$u^2 = \left(\frac{dr}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

and

$$u'^2 = \left(\frac{dr'}{dt'}\right)^2$$

hence

$$\frac{1 - \frac{u'^2}{c^2}}{1 - \frac{u^2}{c^2}} = \frac{c^2 - u'^2}{c^2 - u^2} = \frac{c^2dt'^2 - dx'^2 - dy'^2 - dz'^2}{c^2dt^2 - dx^2 - dy^2 - dz^2} \cdot \frac{dt^2}{dt'^2}$$

$$= \frac{1}{\beta^2\left(1 - \frac{vu_x}{c^2}\right)^2}$$

Finally then

$$\frac{d\tau'}{d\tau} = \frac{1}{\beta\left(1 - \frac{vu_x}{c^2}\right)}$$

or

$$\rho\beta\left(1 - \frac{vu_x}{c^2}\right)d\tau' = \rho d\tau.$$

If then we assume that the charge in the element $d\tau'$ is the same as the charge in $d\tau$, or that

$$\rho' d\tau' = \rho d\tau$$

then

$$\rho' = \rho\beta\left(1 - \frac{vu_x}{c^2}\right)$$

which completes our proof.

In this deduction the relativity hypothesis enters through a Fitzgerald contraction of an electron or an element of it. This furnishes an explanation for the Fitzgerald contraction of a ponderable body, i.e., it is due to the contraction of its ultimate particles.

In the analysis we have also used the fact that

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$$

or that

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2.$$

But these are direct deductions from the Lorentz transformation equations and present no difficulty. If we replace ict by u and ict' by u' the last equation will become

$$x'^2 + y'^2 + z'^2 + u'^2 = x^2 + y^2 + z^2 + u^2$$

or our transformation equations furnish us a basis for a four-dimensional vector analysis of the same character in general as the three-dimensional analysis presented in the introduction. If we call the left member of the last equation the square of the distance from the origin to the point (x', y', z', u') in the frame S', and the right member the square of the distance from the origin to the point (x, y, z, u) in the frame S, we may assert as in the basis for our three-dimensional vector analysis that distances are invariant under a Lorentz transformation.

An event, which is always characterized by a time and a place in this space, is a point in this four-dimensional world of Minkowski, while a continuous train of events will be a "world line," and three-dimensional kinetics becomes four-dimensional statics. Thus the Lorentz transformation correlates space and time in such a way that the relativity of space and time is as evident in the analysis as it is in fact.

17. The Dynamics of the Electron.—In this introduction to the electron theory we are interested in the electromagnetic field in the ether and the motion of the electric

corpuscles through the same medium. But when we wish to write the equations of motion for just a single electron we enter a domain where ordinary dynamics is invalid For one thing the electron in motion is in general acted on by a force due to its own electromagnetic field; this fact must be taken into account in some way. To exhibit this situation we proceed to determine the electromagnetic momentum of an electron.

Suppose the electron is moving with a uniform velocity v along the axis OX; then an observer in the frame S' moving with the velocity v relative to S is at rest relative to the electron To him the field of intensity is a purely electrostatic field, or h' will be zero. To an observer in S the field will be an electromagnetic field with intensities e and h different from zero. Thus the electromagnetic momentum

$$G = -\frac{1}{c^2}\int_\infty s\,d\tau = \frac{1}{4\pi c}\int_\infty e \times h\,d\tau$$

for an electron in motion may be computed from the electrostatic field in S' by using the correlated values given in equations (14 4).

For definiteness we will assume that the electron at rest is a sphere with its charge uniformly distributed over its surface. We have then for this special case

$$e_x = e'_x; \qquad e_y = \beta e'_y; \qquad e_z = \beta e'_z$$

$$h_x = h'_x = 0; \quad h_y = \beta\left(-\frac{v}{c}e'_z\right); \quad h_z = \beta\frac{v}{c}e'_y$$

and there is no difficulty in finding that

$$e \times h = \beta^2\frac{v}{c}(e'^2_y + e'^2_z)i - \beta\frac{v}{c}e'_x e'_y j - \beta\frac{v}{c}e'_z e'_x k.$$

For points symmetrically located with respect to the zx plane e_y has values which are equal but of opposite sign, thus the integral $\int_\infty e'_x e'_y d\tau$, taken throughout all

space, vanishes, and the same thing is true for the integral when e'_z replaces the e'_y. The momentum is thus given by the equation

$$G = \frac{v\beta^2}{4\pi c^2}\int_\infty (e'^2_y + e'^2_z)\frac{d\tau'}{\beta}.$$

In our result we have simply replaced vi by its equal v, and $d\tau$ by its equal $\frac{d\tau'}{\beta}$. It must be noted that $d\tau$ is a volume element in motion relative to the electron, or

$$d\tau' = \beta d\tau.$$

Also

$$\frac{1}{8\pi}\int_\infty e'^2 d\tau' = W'$$

is the electrostatic energy of the electron at rest. Hence

$$\frac{1}{8\pi}\int_\infty (e'^2_y + e'^2_z)d\tau' = \tfrac{2}{3}W'$$

so that

$$G = \frac{4}{3}\frac{W'v}{c^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \tag{17.1}$$

where

$$W' = \frac{1}{8\pi}\int_\infty e'^2 d\tau' = \frac{1}{8\pi}\int_a^\infty \frac{e^2}{r^4}\cdot 4\pi r^2 dr = \frac{e^2}{2a}$$

if the charge is distributed uniformly over the surface of the electron

The total field energy for the moving electron, which is sometimes of interest, is given by the formula

$$E = \frac{1}{8\pi}\int_\infty (e^2 + h^2)d\tau.$$

By substituting the correlated values for e and h it is easily found that

$$E = \frac{W'}{3\beta}\frac{3 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}$$

If the velocity of the electron is changing slowly enough we may regard equation (17.1) as a formula for computing G at any instant as though v were constant for that instant. This is a case of quasi-stationary motion. The approximation is valid, if during a small interval of time, including the instant in question, the change in the velocity during the instant may be neglected.

Since equation

$$-\frac{\partial G}{\partial t} = F$$

gives the force F, acting on the region under consideration, we may say that if the velocity v is constant, the force on the electron is zero. Or an electron moving through the ether with a uniform velocity moves like a material particle under no force. But if v varies in direction or magnitude $\frac{\partial G}{\partial t}$ is no longer zero (17.1), or the electron moves under a force due to its own electromagnetic field.

18. Longitudinal and Transverse Mass.—If the velocity of the electron changes in magnitude and not in direction, then as we have seen (17.1), the electromagnetic momentum G is no longer constant. We will have in this case

$$\frac{dG}{dt} = \frac{d|G|}{d|v|}\frac{dv}{dt}.$$

If we write m' for $\frac{d|G|}{d|v|}$, then $-m'\frac{dv}{dt}$ is the force acting on the electron.

Again, if v varies in direction only, then there is a constant ratio holding between $|G|$ and $|v|$. If we designate this ratio by m'', then

$$G = m''v$$

and

$$\frac{dG}{dt} = m''\frac{dv}{dt}$$

or this force acting on the electron in this case is

$$-m''\frac{dv}{dt}.$$

Usually m' and m'' are called the electromagnetic masses of the electron, the former the longitudinal and the latter the transverse mass. We shall also discriminate between the acceleration in the two cases considered, and write $\frac{dv_l}{dt}$ in the first case where the acceleration is parallel to the direction of motion, and $\frac{dv_t}{dt}$ in the second case where the acceleration is transverse to the direction of motion.

Since the two are mutually perpendicular, we may now combine these two results for the case when v varies in both magnitude and direction. We have then finally

$$-\frac{dG}{dt} = -m'\frac{dv_l}{dt} - m''\frac{dv_t}{dt}$$

for the resultant force acting on the electron.

Thus the force on the electronic charge is not given by the ordinary dynamical law which prescribes that force is proportional to the acceleration. Even the electromagnetic masses are not constants, but rather complicated functions of the velocity. In fact we cannot in general pick a single function of the velocity which when multiplied by the acceleration of the electron will give the force.

If

$$F = \int_\tau \rho\left(e + \frac{1}{c}v \times h\right)d\tau$$

is the total force of electromagnetic origin acting on the electron, and if F' is the resultant of all other forces, and if for convenience we assume that the electron possesses an ordinary mechanical mass m_0, then the equation of motion of the electron will be

$$m_0\frac{dv_l}{dt} + m_0\frac{dv_t}{dt} = -m'\frac{dv_l}{dt} - m''\frac{dv_t}{dt} + F'$$

since

$$F = -\frac{dG}{dt}$$

or

$$(m_0 + m')\frac{dv_l}{dt} + (m_0 + m'')\frac{dv_t}{dt} = F'.$$

But the experiments of Kauffman and others indicate that the total mass of the electron is electromagnetic. If we assume this to be true, and if the total force which holds the electron together and preserves it as an entity has the resultant zero, then the last equation tells us that the electromagnetic forces and the applied forces of non-electromagnetic origin form a system in equilibrium.

Also since

$$G = \frac{2}{3}\frac{e^2}{c^2 a}\frac{v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

for the Lorentz electron, it is a simple exercise to show that

$$m' \equiv m_l = \frac{2e^2}{3ac^2}\beta^3$$

$$m'' \equiv m_t = \frac{2e^2}{3ac^2}\beta$$

and since these are different except when v is zero the force has the directions of the acceleration only when $\frac{dv_l}{dt}$ or $\frac{dv_t}{dt}$ is zero.

18. The Theory of Kauffman's Experiment.—The physical basis for the conclusions of the last article lies in the experiment of Kauffman. To determine the nature of the mass of the electron a stream of electrons from a radioactive substance was passed through a known electric field $\mathbf{e}$, perpendicular to the general direction of motion of the electrons, and their deflections measured. If r is taken as the radius of curvature of the path, and v the speed of the electron, then from elementary mechanics it is known that the acceleration will be $\frac{v^2}{r}$. But $e\mathbf{e}$ is the force acting on the electron if we neglect its own magnetic effect, so that

$$\frac{e\mathbf{e}}{m} = \frac{v^2}{r}.$$

If the electric field be replaced by a magnetic field, the force intensity on the electron will be $\frac{1}{c} v \times h$. If we call r' the radius of curvature in this case we have similarly

$$\frac{ev|h|}{cm} = \frac{v^2}{r'}$$

or

$$\frac{e|h|}{cm} = \frac{v}{r'}$$

In the previous case if r and e are known, $\frac{e}{mv^2}$ is known, also if r' and h are known $\frac{e}{mv}$ is known. From the values of $\frac{e}{mv}$ and $\frac{e}{mv^2}$, v and $\frac{e}{m}$ may be computed. The velocities of the electrons were found to range from .6 to .9 the velocity of light, while $\frac{e}{m}$ decreased with the velocity. This decrease can be attributed only to an increase in mass, in fact, the decrease in $\frac{e}{m}$ was so marked that the mass m would seem to be entirely electromagnetic, which agrees well with the preceding article where the longitudinal and transverse masses were found to be functions of the velocity.

This is one of the outstanding results of modern physics. This, together with the ideas of the previous article, makes it clear that the relativity hypothesis changes materially the character of particle dynamics As has been stated before, the most general problem of electrodymanics is to find the motion and distribution of the charges when the state of the electromagnetic field is known at a given instant. To complete the Lorentz field equations in the case of a single electron we were able to write out its equations of motion, approximately, using the electromagnetic momentum. This depended on assumptions about the shape of the electron, the distribution of its charge, and

some mechanism for maintaining it as an entity. The ether which supports electromagnetic phenomena also prevents the eruption of the electron, or the force between its elements obeys a law entirely different from the usual electrostatic law. To write out the equations of motion in the more general case where large numbers of corpuscles are involved would be practically impossible; but this is a statistical or an average value problem, and requires other methods. In fact, the applicability of the detailed analysis of the electron theory would seem at best to be very limited where intro-atomic forces are involved though it is known that the Priestley or Coulomb law holds down to a very small fraction of a centimeter. In the study of the atom, a microscopic analysis is avoided and only observable magnitudes enter the quantum mechanical analysis of this important problem of modern physics.

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